Vedic Mathematic

Bindu

Bhaktivedanta Academy

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1 Arithmetic (aṅkagaṇitam)

There are three branches of mathematics; the first branch is arithmetic, which deals with numbers.

The number 1, the first number, represents unity . Unity and wholeness are everywhere.

The system of Vedic Mathematics is based on 16 formulas or *sūtras*. We will start with the following *sūtra: ekādhikena pūrveṇa*, which means "By One More Than the One Before"

All whole numbers greater than the number 1 can be generated by using the formula, "By One More Than the One Before": 2 is one more than l, 3 is one more than 2, and so on. We will see many other applications of this simple formula as we go on. From this *sūtra* we can see that there is only one number 1. That is Kṛṣṇa. Everything else is His expansion. Similarly all the other numbers are expansions of number 1 .

Whole numbers are also called natural numbers.
Each number differs from all other numbers. Like *avatāras*, each number has its own special and unique properties.

* Write the numbers 1 to 10 vertically down the left side of your page leaving about two lines of space between each number.

* For each number, discuss where you have seen this number illustrated with a partner. For example, for the number 7 you may think of the 7 days of the week or the 7 colors of the rainbow. Write your observations under each number in your notebook, utilizing the two lines of space left for this purpose.

* Discuss a few properties of different numbers.
For example, how can you distinguish between odd and even numbers, what is special about the number 1 , what is special about the number 10 ?

Think of as many possibilities as you can.

Exercise 1

Write down the number that is one more than:

a) 6 b) 17 c) 9 d) 19 e) 21 f) 30 g) 99 h) 199 i) 300 j) 401 k) 9,999

l) 309 m) 6,789 n) 7,829,009 o) 9,699 p) 9,998 q) 78,979

There is also a *sūtra* called *'ekānuneṇa pūrvena'*.

* What do you think this means? (Look at your list of *sūtras*)

Exercise 2

Go back to exercise 1, but this time write down the number that is one less than the number given.

Exercise 3

Ask your teacher for more exercises.

2 Seed Digit (Bījāṅkam) and the 9-Point Circle

The *Bījāṅkam* is actually the sum of a number's digits. The word **digit** refers to all numbers from 0 through 9 (0,1,2,3,4,5,6,7,8, and 9)-. The word **sum** means the total you get when adding two or more numbers. For example, the sum of 2+3 is 5.

The *bījāṅkam* of a number is found by adding the digits of a number. In numerology, we will need to know how to find the *bījāṅkam* of numbers to uncover the characteristics of a particular number.

Example 1

Find the *bījāṅkam* of a) 17 b) 401.

a) For 17 we get 1 + 7 = 8, b) For 401 we get 4 + 0 + 1 = 5.

Exercise 1

Find the *bījāṅkam* of the following numbers:

a) 16 b) 27 c) 32 d) 203 e) 423 f) 30103

g) The *bījāṅkam* of a 2-figure number is 8 and the figures are the same. What is the number?

h) The *bījāṅkam* of a 2-figure number is 9 and the first figure is twice the second. What is it?

Example 2

Find the *bījāṅkam* for 761.

For 761 we get 7 + 6 + 1 = 14 and as 14 is a 2-figure number we add the figures in 14 to get 1 + 4 = 5.

So the *bījāṅkam* of 761 is 5.

The *bījāṅkam* is found by adding the digits in a number, and adding again if necessary.

This means that any natural number of any size can be reduced to a single digit: just add all digits, and if you get a number with two or more digits, add again until only one digit remains.

Exercise 2

Find the *bījāṅkam* of:

a) 37 b) 461 c) 796 d) 6437

e) 3541 f) 5621 g) 4978272 h) 673982741

The Number Nine

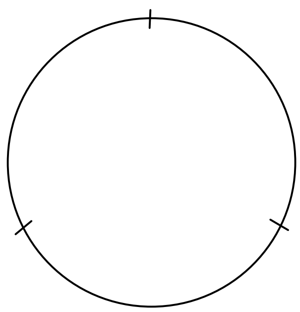
The number 9 is very important in the Vedic system. As we will see, it has many remarkable properties, which make it very interesting and very useful.

The 9-Point Circle

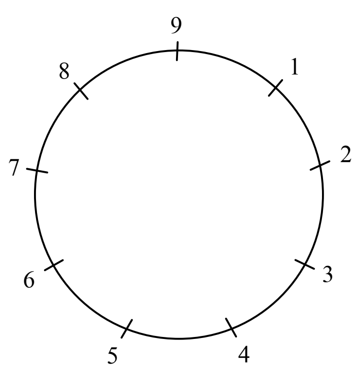
The 9-point circle is a circle with an edge that is divided into 9 equal parts.

*First draw a circle, with your compass set on a radius of about 4 cm. Leave plenty of space around your circle.

* As accurately as you can, divide your circle into thirds like the circle drawn here. Ask your guide for a hint on how to do this.



* Now divide each third into three and number the circle as shown.



This is known as the 9-point circle. We are now going to continue numbering around the circle.

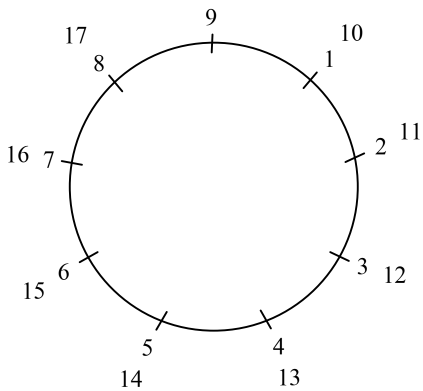
* Where shall we put the number 10?

It seems reasonable to put 10 after 9, where we have already placed the number 1.

We can continue numbering around the circle as shown below.

* Where will the number 19 go?

* Continue numbering around the circle up to 30.



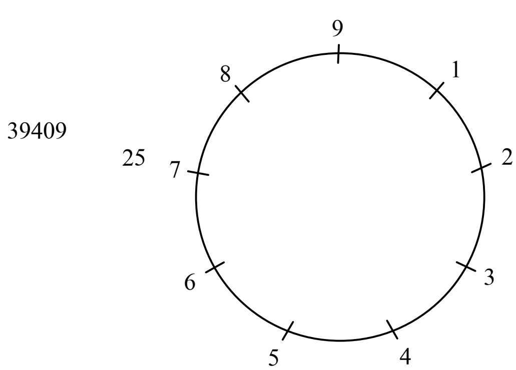
1) Write down all the numbers you get on the `branch' that begins with 1, (1,10,19, etc.)
2) What do you notice about the *bījāṅkam of these numbers*?
3) Predict the next 3 numbers on this branch.
4) What are their *bījāṅkam*?
5) What can you say about the *bījāṅkam* of all the numbers on this branch?
6) Do the same thing for two other branches. What do you notice?
7) Starting at 3 what do you have to add on to get back to the 3 branch?
8) What would you have to add on to any one number to get back to the branch you started at?

Adding 9 to a number does not affect its *bījāṅkam*:
For example, 7, 70, 79, 97, 979 all have a *bījāṅkam* of 7.

Example 3

Find the *bījāṅkam* of 39409.

We can cast out the nines and just add up the 3 and 4. So the *bījāṅkam* is 7. Or using the longer method, we add all the digits: 3 + 9 + 4 + 0 + 9 = 25 = 7 again.



Exercise 3

Try working out the *bījāṅkam* of the following numbers from beginning to end and then from end to beginning to see if you get the same answer. Show your working clearly.

a) 39 b) 93 c) 993 d) 9993 e) 9329 f) 941992

Looking again at the 9-point circle, if we count backwards around the circle we see that since 0 comes before 1 it is logical to put zero at the same place as 9. In terms of *bījāṅkam*, 9 and 0 are equal. This also explains why we can cast out nines.

This casting out the nines can be used in another way:

Any group of figures in a number that add up to 9 can also be cast out.

Example 4

Find the *bījāṅkam* of a) 24701 b) 21035

a) In 24701 we see 2 and 7, which add up to 9. We can therefore cast out 2 and 7 and add up only the other figures. 4 + 0 + 1 = 5.

b) In 21035 we can see that 1, 3 and 5 add up to 9 so we cast these numbers out. The sum of the remaining figures is 2 so the answer is 2.

Exercise 4

Calculate the *bījāṅkam* of the following:

a) 39 b) 94 c) 95 d) 995 e) 9995 f) 634
g) 724 h) 534 i) 627 j) 9653 k) 36247 l) 9821736

Exercise 5

Ask your teacher for more exercises.

3 The Vedic Numerical Code (kaṭapayādi)

All scientific subject matter in the *śāstras* is explained through verses. Thus, numerical values are represented by *Devanāgarī* letters to facilitate the recording of the arguments, sometimes to conceal the real subject, but mostly to help the student to learn. To memorize a meaningful verse is much easier than to memorize plain numbers. This is why many theological, philosophical, medical, and astronomical treatises, as well as huge dictionaries have been written in Sanskrit verse. The authors and/or compilers of these works used verse, *sūtras*, and codes to lighten the learner'sburden and to facilitate the work by explaining scientific and mathematical material simply, through verses.

The table below shows the codes:

ka	ṭa	pa	ya	1
kha	ṭha	pha	ra	2
ga	ḍa	ba	la	3
gha	ḍha	bha	va	4
ṅa	ṇa	ma	śa	5
ca	ta		ṣa	6
cha	tha		sa	7
ja	da		ha	8
jha	dha			9

ña, na, kṣa and vowels are considered as zero or not considered at all.

Example 1

kevalaiḥ saptakaṁ guṇyāt (one of the *subsūtras*)

kevalaiḥ = 143

saptakam = $\frac{1}{7}$

i.e. $\frac{1}{7} = \frac{143 \times 999}{999 \;\; 999} = \frac{142857}{999999} = 0.14285\dot{7}$

It is actually a decimal value of $\frac{1}{7}$.

Example 2

kalau kṣdrasasaiḥ

kalau = 13

kṣdrasasaiḥ = 077

i.e. $\frac{1}{13} = \frac{077 \times 999}{999 \;\; 999} = \frac{076923}{999999} = 0.07692\dot{3}$

Decimal value of $\frac{1}{13}$.

Example 3

kaṁsekṣāmadāhakhalairmalaiḥ

kaṁse = 17
kṣāmadāhakhalairmalaiḥ = 05882353

i.e. value of $\frac{1}{17}$

Example 4

gopī-bhāgyamadhuvrāta-śṛṅgiśodadhisandhiga /
khalajīvitakhātāva galahālārasaṁghara //

i.e. $\frac{\pi}{10} = .3141592653589793/2384626433832792$

Your Psychic And Destiny Number

*Write down your full date of birth (e.g. 16-9-1993) and find its *bījāṅkam*. That *bījāṅkam* is your destiny number. To get your psychic number, simply find the *bījāṅkam* of only the **day** you were born. We will discuss this subject more when we come to numerology.

*You can also find the *bījāṅkam* of your phone number and of the time right now.

Bījāṅkam Problems

Exercise 5
Find a two-figure number with a *bījāṅkam* of
a) 6 where the figures are equal
b) 6 where the 1st figure is double the 2nd
c) 7 where the difference between the figures is 3 (two answers)
d) 7 where one figure is a 4 (two answers)
e) 6 where both figures are odd (three answers)
f) 5 where the figures are consecutive numbers # (two answers)
g) 9 where the figures are consecutive (two answers)
h) 9 where one figure is double the other (two answers)
i) 8 where the number is below 20
j) 1 where the first figure is a 4

Consecutive means one after the other (e.g. 6 and 7 or 7 and 6 are consecutive.

Exercise 6

Find a two-figure number with a

a) *bījāṅkam* of 5 that ends in 2
b) *bījāṅkam* of 8 that is between 50 and 70 (two answers)
c) *bījāṅkam* of 7 where the figures are equal
d) *bījāṅkam* of 2 and one of the figures is a 3 (two answers)
e) *bījāṅkam* of 1 that ends in 3
f) *bījāṅkam* of 3 that ends in 8
g) *bījāṅkam* of 1 where the figures are equal
h) *bījāṅkam* of 2 that ends in 7
i) *bījāṅkam* of 6 where one figure is a 2 (two answers)
j) *bījāṅkam* of 3 which is greater than 80 (two answers)
k) *bījāṅkam* of 2 where one figure is a 4 (two answers)
l) *bījāṅkam* of 5 where both figures are odd (three answers)
m) *bījāṅkam* of 2 where the number is even (five answers)
n) *bījāṅkam* of 8 where the number is odd (five answers)
o) *bījāṅkam* of 3 where the figures differ by 6 (two answers)

Exercise 7

Ask your teacher for more exercises.

4 Large Numbers

In a large number like 1234, the value of each figure depends on its position:

thousands	hundreds	tens	units
1	2	3	4

The 4, being at the extreme right is 4 units;
the 3 is in the tens position, and so it represents 3 tens, or 30;
the 2 is 2 hundreds;
and the 1 is 1 thousand.

If a particular column is empty, we write 0 in that column.
For example, 3030 is three thousand and thirty.

Our number system uses the number 10 as a base.
We will call the numbers 10, 100, 1000 etc. base numbers.

Restructuring Numbers

We need to be flexible about the way numbers are shown in this system.
Sometimes it is necessary to restructure a number.

Example 1
In the number 33 we have 3 tens and 3 units

tens	units		tens	units
3	3	=	2	13

Here 33 is rewritten as 2 tens and 13 units.

tens	units		tens	units
3	3	=	1	23

and here it becomes 1 ten and 23 units.

Example 2
It is also easy to convert a number like

tens	units
4	16

into the simpler form by, in this case, adding the 1 to the 4 to get 56.

Example 3

Similarly

hundreds	tens	units		hundreds	tens	units
4	3	2	=	4	2	12

Also

hundreds	tens	units		hundreds	tens	units
4	3	2	=	3	13	2

Exercise 1

Put into their simplest form (h/t/u stands for hundreds, tens and units):

a)

h	t	u
1	4	13

b)

h	t	u
	5	23

c)

h	t	u
5	1	22

d)

h	t	u
	9	15

e)

H	t	u
3	14	8

f)

h	t	u
1	25	7

g)

h	t	u
3	13	13

h)

h	t	u
9	9	10

Copy the following and insert the correct number where there is a question mark:

i)

		t	u		t	u		t	u
44	=	3	?	=	2	?	=	1	?

j)

		h	t	u		h	t	u		h	t	u		h	t	u
555	=	5	4	?	=	5	?	25	=	4	?	5	=	4	?	25

k)

		h	t	u		h	t	u		h	t	u		h	t	u
741	=	6	?	1	=	5	?	1	=	7	2	?	=	5	?	21

Reading And Writing Large Numbers

A thousand is ten hundreds. It is written 1 000, 1,000 or 1000.
If you arranged ten bricks in a row and then put ten of these rows together to make a square, you would have 100 bricks.
If now you had ten identical layers of 100 bricks on top of each other you would have ten hundreds or 1000 bricks.

Example 4

The number 54,321 represents fifty four thousand, three hundred, and twenty one.
"54" is pronounced first, then "thousand" (where the comma is), then "321".

Exercise 2

Write the following in words; try to get the spelling right:

a) 76,462 b) 18,703 c) 6,480 d) 90,008

e) 111,234 f) 41,070 g) 3,800 h) 309,511

i) 88,055 j) 123,456 k) 30,000 l) 120,300

Sometimes you may see large numbers written out without the commas. But since the comma is always three figures from the end, it is easy to see where the comma should be.
Sometimes a space is left instead of a comma.

So you might see 76 000 or 76,000 or just 76000. Any of these forms might be used in our class.

Now can we do the opposite? Given a number in words, can we write it in figures?

Example 5

Seventeen thousand nine hundred and forty six would be written as 17,946. Note that the word "thousand" in this example shows you where the comma goes.

Example 6

Seven hundred eighty three thousand and sixty six.
First write down seven hundred and eighty three, 783, then the comma, then 66.
This gives us 783,066.

Carefully note the extra 0; this 0 must be inserted. Without this 0 the number would be 78366, which says seventy eight thousand three hundred and sixty six; this is a different number.

There must always be three figures after the thousands and this may involve adding 0 or 00 or 000 depending on the value of the number.

Exercise 3

Write the following in figures:
a) Seven thousand eight hundred and twenty
b) Thirty three thousand four hundred and twelve.
c) Fifty one thousand eight hundred and ninety.
d) Thirteen thousand six hundred and seventy seven.
e) Thirty four thousand eight hundred and four.
f) Three hundred and forty three thousand seven hundred and eleven.
g) Four hundred and ten thousand two hundred and eighty.
h) Eight hundred and eighteen thousand and seventy four.
i) Fifty seven thousand and ninety nine.
j) Eighty thousand nine hundred and nine
k) Three hundred thousand seven hundred and forty six.
1) Two hundred and twenty thousand and one.
m) Ninety thousand.
n) Four hundred and four thousand three hundred and three.

Millions

This system can be extended to include millions, billions, and so on.

A million is a thousand thousands. It is written in figures like 1,000,000 or 1000,000. These are huge numbers.

Can you think of a way of explaining 1 million, like we had for 1 thousand?

Take one sheet of a square paper and look at it. Find out how many of the smallest squares are on the whole sheet. Look for an easy way of doing this.

*How many sheets of square paper would you need to have 1 million small squares?

Example 7

Write 23,456,789 in words.

Because there are two commas, we know we have millions here (23 million).
The middle part contains the thousands, so we have 456 thousands.

So we can write
twenty three million four hundred and fifty six thousand seven hundred and eighty nine. Clearly we are dividing this number into three parts.

The *sūtra* we have been using here is *lopana- sthāpanābhyām* which means "By Alternate Elimination and Retention" as we work with each of the three parts of the number individually, one part at a time; when we look at the millions we ignore the rest of the number, when we look at the thousands, we ignore the first and last parts, and so on.

Example 8

Write in words 103,040,005.

This says one hundred and three million forty thousand and five.

Example 9

Write in words 17,000,800.

Seventeen million eight hundred (there are no thousands here).

Exercise 4

Write in words:

a) 6,202,304 b) 10,005,070 c) 85, 085,200 d) 111,900,007

e) 108,108,108 f) 99,000,333 g) 2,020, 020 h) 999,001,070

Exercise 5

Write in numbers :
a) Eight million eight thousand three hundred and four.
b) Ninety million eight hundred and seven thousand and one.
c) Three hundred million four hundred thousand and seventy seven.
d) Twelve million five hundred and sixty five thousand six hundred and fifty six.
e) Four million and forty.
f) Four hundred and thirty two million.

According to Vedic mathematical calculations, the following enumeration system is used:

1	*eka* (units)
10	*daśa* (tens)
100	*śata* (hundreds)
1,000	*sahasra* (thousand)
10,000	*ayuta* is ten thousand
100,000	*lakṣa* is hundred thousand
1,000,000	*niyuta* or one million is ten times *lakṣa*
10,000,000	*koṭi* is ten times *niyuta*
100,000,000	*arbuda* is ten times *koṭi*
1,000,000,000	*vṛnda* or one billion is ten times *arbuda*

Exercise 6

Ask your teacher for more exercises.

5 Bījāṅkam Check

Now that we know how to calculate *bījāṅkam*, we can look at a few useful applications. One very common use of *bījāṅkam* is to determine whether or not a number is divisible by another number. We will examine this use in a later chapter. In this chapter, we will learn a method of checking the answers to addition or subtraction problems . We will also learn a new method of subtraction.

Addition (saṅkalanam)

Page: 20
[0]Saṅkalanam – the action of putting parts & parcels together.

sam + *kalanam*; *sam* means put together; *kāla* means parts and *ana* + *am* means the action of

Addition From Right To Left

While addition is the simplest operation, when we have to add several numbers, addition can become difficult. For this reason, we use a simple method called the dot method (*bindvaṅkanam*).

Before we employ the dot method, we need to revise our method for writing down simple addition from right to left. In a later chapter we will learn how to do both, written and mental calculations from left to right.

Example 1

Find 3451 + 432

	3	4	5	1
+		4	3	2
	3	8	8	3

We line the numbers up with the units directly under the units. There are no remainders, so we simply add in each column without putting dots: 1 + 2 = 3; 5 + 3 = 8; 4 + 4 = 8; 3 + 0 = 3.

Exercise 1

Solve the following addition sums, setting them out in a similar way to the example.

a) 43 + 23

b) 21 + 34 + 42

c) 123 + 201

d) 401 + 121 + 77

e) 4321 + 511

f) 2102 + 127 + 330

g) 33 + 21   h) 111 + 212   i) 203 + 21   j) 1414 + 123 + 212   l) 234120 + 54302 + 43

Sums involving 'carries'

Example 2

$$\begin{array}{ccc} & 5 & \dot{6} \\ + & 1 & 7 \\ \hline & & 3 \\ \hline \end{array}$$

In this example, when we add 7 and 6. We get 13, which is 10 + 3, so we write down 3 in the units and put dot above 6.

$$\begin{array}{ccc} & 5 & \dot{6} \\ + & 1 & 7 \\ \hline & 7 & 3 \\ \hline \end{array}$$

Next, we add 1 + 5 = 6, but before you finish, count the number of dots in the previous column.

Example 3

$$\begin{array}{ccc} & \dot{9} & \dot{7} \\ + & 8 & 8 \\ \hline 1 & 8 & 5 \\ \hline \end{array}$$

Similarly, when we add 7 and 8 we get 15. So we write down 5 and put one dot above 7. Next we add 8 + 9 and get 17. Then we look in the previous column where we find one dot. So we add 1 to 7 and in the second column we put the total, which is 8. Then finally we have one dot above 9, which we place in the column furthest to the left. The result is 185.

Example 4

$$\begin{array}{ccc} & \dot{5} & \dot{6} \\ & \dot{8} & \dot{7} \\ + & 7 & 8 \\ \hline 2 & 2 & 1 \\ \hline \end{array}$$

8+7=15, so put a dot above 7 and continue with 5 only; 5+6=11, so put a dot above 6 and write down 1 as the first digit of the answer; then in next column 7+8=15, so put a dot above 8 and continue with 5. 5+5=10. Now you count the dots from the previous column. We see 2 dots there so 10+2=12, so write down 2, but we have two more dots in this column, so that you put as 2 in the next one.

Exercise 2

Add the following. Leave plenty of room on either side of the sum for future calculations.

a) $\begin{array}{ccc} & 4 & 3 \\ + & 2 & 8 \\ \hline \\ \hline \end{array}$ b) $\begin{array}{ccc} & 3 & 7 \\ + & 2 & 8 \\ \hline \\ \hline \end{array}$ c) $\begin{array}{ccc} & 5 & 6 \\ + & 2 & 9 \\ \hline \\ \hline \end{array}$ d) $\begin{array}{cccc} & 1 & 3 & 1 \\ + & 2 & 7 & 1 \\ \hline \\ \hline \end{array}$ e) $\begin{array}{cccc} & 8 & 6 & 7 \\ + & & 6 & 7 \\ \hline \\ \hline \end{array}$

f) 83 + 77 g) 38 + 48 h) 74 + 69 i) 777 + 136 j) 2563 + 1723

The Bījāṅkam Check

Having solved a few simple addition problems, we can now learn how to perform a simple check using *bījāṅkam* to see if the answers are correct.

The corresponding Vedic *sūtra* for this check is *guṇita-samucchayaḥ samucchaya-guṇitaḥ*, which means, "The Product of the Sum is the Sum of the Products." The product is what you get when you multiply numbers together. According to this *sūtra, the bījāṅkam* of the sum is the sum of the *bījāṅkam*.

Example 4

Add 32 and 12 and check the answer using *bījāṅkam*.

$$\begin{array}{rrr} & 3 & 2 \\ + & 1 & 2 \\ \hline & 4 & 4 \\ \hline \end{array} \qquad \begin{array}{rr} & 5 \\ + & 3 \\ \hline & 8 \\ \hline \end{array}$$

Here the digit sum of 32 is 5 (3 + 2 = 5) and the digit sum of 21 is 3. The sum (the total) of the digit sums is 5 + 3 = 8. If we have done the sum correctly, the digit sum of the answer should be also 8. 4 + 4 = 8; so according to this check the answer is probably correct.

Example 5

Add 365 and 208 and check the answer.

$$\begin{array}{rrrr} & 3 & 6 & \dot{5} \\ + & 2 & 0 & 8 \\ \hline & 5 & 7 & 3 \\ \hline \end{array} \qquad \begin{array}{rr} & 5 \\ + & 1 \\ \hline & 6 \\ \hline \end{array}$$

We get 573 for the answer. The digit sums of 365 and 208 are 5 and 1, respectively. 5+1 = 6. Finally, because the digit sum of 573 is also 6, the answer is confirmed.

Example 6

Add 77 and 124 and check.

$$\begin{array}{rrrr} & & \dot{7} & \dot{7} \\ + & 1 & 2 & 4 \\ \hline & 2 & 0 & 1 \\ \hline \end{array} \qquad \begin{array}{rr} & 5 \\ + & 7 \\ \hline & 3 \\ \hline \end{array}$$

This is similar except that when we add 5 and 7 we do not put down 12. Instead, we put down 3, the digit sum of 12.

Now go back to exercise 2 and check your answers using the '*bījāṅkam* check'. If you find any mistakes, go back and correct them! Use the system of writing *bījāṅkam* check as shown above.

Exercise 3

Add the following and check your answers using the *bījāṅkam*:

a)	b)	c)	d)	e)	f)
3 5	5 6	3 5	5 2	4 5 6	1 8 8
+ 4 7	+ 2 7	+ 5 9	+ 2 4	+ 3 3 3	+ 2 7 7
____	____	____	____	____	____

Exercise 4

Which of the sums below are wrong? Use the *bījāṅkam* check to find out.

a)	b)	c)	d)	e)	f)	g)
4 3	9 1	5 8	5 5	4 6	8 8	4 3 8
7 5+	1 9+	7 6+	3 9+	4 7+	7 7+	2 3 8+
1 1 8	1 1 0	1 2 4	9 4	9 4	1 6 5	6 7 6

h)	i)	j)	k)	l)	m)	n)
5 1 6	4 4 4	4 6	9 1 1	7 9	5 3	8 3
6 1 5+	8 7 8+	7 1	1 1 9+	8 9+	6 7 6+	3 8+
1 1 3 1	1 3 1 2	3 8+	1 0 3 0	1 6 8	7 2 9	1 3 1
		1 5 4				

Exercise 5:

Get more exercises from your teacher.

Example 7

Check the following sum:		The check is:		
	2 7 9		9	which confirms the answer.
	1 2 1+		4+	
	4 9 0		4	

However if you check the addition of the original sum you will find that it is incorrect! This shows that *bījāṅkam* method does not always catch errors. We will look at checking methods later on.

Subtraction (vyavakalanam)

Page: 23
[0]vyavakalanam – the action of separating or dividing parts & parcels from each other

vyava + *kalanam* → *vi* + *ava* – pull apart; *kāla* – parts; *ana + am* – the action of

Subtraction from right to left

For subtracting small numbers from larger numbers, the common method is sufficient (e.g. 9 – 7 = 2); this method is actually addition as it gives us the number that must be added to 7 to get 9. The problem arises when we subtract bigger numbers from smaller numbers. In this case we can use the "borrowing" method.

Example 8

Find the answer for 35567 – 11828.

Set the sum out as usual:

$$\begin{array}{r} 3\,4\,5\,6\,7 \\ -\,1\,5\,8\,2\,5 \\ \hline 2 \\ \hline \end{array}$$

Then starting on the right, subtract in each column. First, 7 – 5 = 2, so put down 2.

$$\begin{array}{r} 3\,4\,5\,6\,7 \\ -\,1\,5\,8\,2\,5 \\ \hline 4\,2 \\ \hline \end{array}$$

In the next column, we have 6 − 2 = 4, so we put down 4 and continue with the next column.

$$\begin{array}{r} 3\,4\,\dot{5}\,6\,7 \\ -\,1\,5\,8\,2\,5 \\ \hline 7\,4\,2 \\ \hline \end{array}$$

Then in the next column, we have 5 - 8. So for this we have to borrow and calculate 15 - 8 = 7. We put down 7, but we also put a dot above 5.

$$\begin{array}{r} 3\,\dot{4}\,\dot{5}\,6\,7 \\ -\,1\,5\,8\,2\,5 \\ \hline 8\,7\,4\,2 \\ \hline \end{array}$$

In the next step, we add 5 with the dot from the previous column and get 6. Then we have 4 - 6. To solve this, we borrow 1 and get 14 - 6 = 8. We put down 8, and put a dot above 4.

$$\begin{array}{r} 3\,\dot{4}\,\dot{5}\,6\,7 \\ -\,1\,5\,8\,2\,5 \\ \hline 1\,8\,7\,4\,2 \\ \hline \end{array}$$

Finally we add 1 and the dot in the previous column and get 3 – 2 = 1. Put down 1. The result is 18742.

Exercise 5

Subtract the following from left to right. Leave some space to the right of each sum so that you can check them later.

a) $\begin{array}{r} 4\;4\;4 \\ 1\;8\;3\;- \\ \hline \\ \hline \end{array}$ b) $\begin{array}{r} 6\;3 \\ 2\;8\;- \\ \hline \\ \hline \end{array}$ c) $\begin{array}{r} 8\;1\;3 \\ 3\;4\;5\;- \\ \hline \\ \hline \end{array}$ d) $\begin{array}{r} 6\;9\;5 \\ 3\;6\;8\;- \\ \hline \\ \hline \end{array}$ e) $\begin{array}{r} 7\;6\;5 \\ 3\;6\;9\;- \\ \hline \\ \hline \end{array}$ f) $\begin{array}{r} 5\;0\;4 \\ 2\;7\;5\;- \\ \hline \\ \hline \end{array}$

g) $\begin{array}{r} 5\;1 \\ 3\;8\;- \\ \hline \\ \hline \end{array}$ h) $\begin{array}{r} 3\;4\;5\;6 \\ 2\;8\;1\;- \\ \hline \\ \hline \end{array}$ i) $\begin{array}{r} 7\;1\;1\;7 \\ 1\;7\;7\;1\;- \\ \hline \\ \hline \end{array}$ j) $\begin{array}{r} 8\;0\;0\;8 \\ 3\;8\;3\;9\;- \\ \hline \\ \hline \end{array}$ k) $\begin{array}{r} 5\;1\;6\;1 \\ 1\;8\;3\;8\;- \\ \hline \\ \hline \end{array}$ l) $\begin{array}{r} 9\;8\;7\;6 \\ 6\;7\;8\;9\;- \\ \hline \\ \hline \end{array}$

m) $\begin{array}{r} 6\;3\;6\;3 \\ 3\;3\;8\;8\;- \\ \hline \\ \hline \end{array}$ n) $\begin{array}{r} 5\;1\;0\;1\;5 \\ 2\;7\;9\;8\;6\;- \\ \hline \\ \hline \end{array}$ o) $\begin{array}{r} 1\;4\;2\;8\;5 \\ 7\;1\;4\;8\;- \\ \hline \\ \hline \end{array}$ p) $\begin{array}{r} 8\;8\;1\;8\;8 \\ 1\;9\;1\;8\;9\;- \\ \hline \\ \hline \end{array}$ q) $\begin{array}{r} 9\;6\;3\;0\;3\;6\;9 \\ 3\;6\;9\;0\;9\;6\;3\;- \\ \hline \\ \hline \end{array}$

Checking Subtraction Sums

Example 10

Find 69 – 23 and check the answer.

$$\begin{array}{r} 6\;9 \\ 2\;3- \\ \hline 4\;6 \\ \hline \end{array} \qquad \begin{array}{r} 6 \\ 5- \\ \hline 1 \\ \hline \end{array}$$

The answer is 46.
The *bījāṅkam* of 69, 23 and 46 are 6, 5 and 1.
And 6 – 5 = 1, which confirms the answer because here we subtract the digit sums.

Example 11

$$\begin{array}{r} 5\;6 \\ 2\;9- \\ \hline 2\;7 \\ \hline \end{array} \qquad \begin{array}{r} 2 \\ 2- \\ \hline 9 \\ \hline \end{array}$$

In this example, the *bījāṅkam* of both 56 and 29 is 2 and the *bījāṅkam* of 27 is 9.
We get 2 – 2 = 0, but we have already seen that 9 and 0 are equivalent as *bījāṅkam*, so the answer is confirmed.

Alternatively, we can add 9 to the upper 2 before subtracting the other 2 from it: 11 – 2 = 9. This is because adding 9 to any number takes you round the circle and back to where you started: 2 and 11 are equivalent in terms of *bījāṅkam*.

Example 12

$$\begin{array}{r} 6\;7\;9 \\ 2\;3\;3- \\ \hline 4\;4\;6 \\ \hline \end{array} \qquad \begin{array}{r} 4 \\ 8- \\ \hline 5 \\ \hline \end{array}$$

Here we have 4 – 8 in the *bījāṅkam* check, so we simply add 9 to the upper figure (the 4) and continue: 13 – 8 = 5, which is also the *bījāṅkam* of 446.

Exercise 6

a) Check your answers to Exercise 5 by using the *bījāṅkam* check.

b) Which of the following sums are wrong? Use the *bījāṅkam* check to find out.

$$\text{a)}\begin{array}{r} 4\;8 \\ 2\;3- \\ \hline 2\;5 \\ \hline \end{array} \quad \text{b)}\begin{array}{r} 4\;7 \\ 2\;5- \\ \hline 2\;2 \\ \hline \end{array} \quad \text{c)}\begin{array}{r} 7\;8 \\ 6\;9- \\ \hline 1\;2 \\ \hline \end{array} \quad \text{d)}\begin{array}{r} 1\;4\;7 \\ 6\;9- \\ \hline 7\;8 \\ \hline \end{array} \quad \text{e)}\begin{array}{r} 2\;3\;8 \\ 4\;7- \\ \hline 1\;8\;1 \\ \hline \end{array} \quad \text{f)}\begin{array}{r} 8\;7\;6 \\ 3\;8\;9- \\ \hline 4\;8\;7 \\ \hline \end{array}$$

Exercise 7

Ask your teacher for more exercises.

6 Number Nine

In our number system the number 9 is the largest digit.
The number nine also has many other remarkable properties, which make it extremely useful.
We have already seen that the number 9 can be used in finding *bījāṅkam*, and that the *bījāṅkam* of a number is unchanged if 9 is added to it or subtracted from it.

Now look at the 9- multiplication table:

9	x	1	=	9
9	x	2	=	18
9	x	3	=	27
9	x	4	=	36
9	x	5	=	45
9	x	6	=	54
9	x	7	=	63
9	x	8	=	72
9	x	9	=	81
9	x	10	=	90
9	x	11	=	99
9	x	12	=	108

If you look at the answers you will see that in every case the *bījāṅkam* is 9.

You may also see that if you read the answers as two columns the left column goes up (By One More than the One Before [I, 2, 3, . . .]) and the right column goes down (By One Less than the One Before [9, 8, 7,…………..]).

Next, look at subtraction: choose a 2-figure number in which the first figure is greater than the second. Reverse the figures and subtract from the original number.

For example, 83 is a number in which the first figure is greater than the second.

Reversing the figures gives 38 which we subtract from 83:

8 3
3 8 –
——

——

1) Finish this sum and then find the *bījāṅkam* of the answer.
• Then, choose a 2-figure number of your own and do a similar subtraction.

2) What is the *bījāṅkam* of your answer?

3) Get the sum of the following problem:

```
 8 3 1
 3 1 8 –
 ______

 ______
```

4) What is the *bījāṅkam* of the answer?

5) Solve:

```
 7 5 6 3
 3 7 6 5 –
 ________

 ________
```

and the *bījāṅkam* of the answer.

In the above two sums the figures are not reversed, but rearranged.

• Choose a number as small or large as you like, rearrange the figures (but make sure that the first figure is smaller than the first figure of the original number), subtract and find the *bījāṅkam* of the answer.

You should find that it is 9 whatever number you choose.

6) Can you explain why this happens?

By Addition And By Subtraction
saṅkalana-vyavakalanābhyām

This *sūtra* is very useful when adding or subtracting numbers which end in 9.

Suppose we want to add 9 to another number, say 44. We could count 44,45,46,47......etc. 9 times, being careful not to make a mistake.
• Can you think of another way of solving this problem?

If we think of 9 as being one less than 10 we could solve this problem using the *sūtra saṅkalana-vyavakalanābhyām*, which means, “By Addition and by Subtraction.” In other words, we could add on 10 and then subtract 1.Adding 10 to 44 gives 54, and taking 1 off gives 53.

Perhaps you would have done it this way or you may have just known the answer, like knowing your multiplication tables. Still, we can use this idea for more complicated problems.

Example 1

Find a) 55 + 29 b) 222 + 59 c) 345 + 199

Each of these can be done in an easy way using the above formula.

a) We could add 30 to 50 and take 1 off. So we can write the answer straight down:
55 + 29 = 84

b) Here we add 60 and take 1 off: 222 + 59 = 281.

c) 199 is 1 below 200 so we can add 200 to 345 and take 1 off: 345 + 199 = 544.

Example 2

Find a) 55 – 29 b) 222 – 59 c) 345 – 199

a) Since 29 is 1 below 30 we can subtract 30 and add 1 back on: 55 – 29 = 26.

b) Subtract 60 and put 1 back: 222 – 59 = 163.

c) Subtract 200 and add 1: 345 – 199 = 146.

Example 3

Find a) 66 + 28 b) 444 – 297

The same method can be applied here.

a) We can add 30 to 66 and take 2 away: 66 + 28 = 94.

b) Here we can subtract 300 and add 3 back on: 444 – 297 = 147.

Exercise 1

Solve the following problems mentally:

Add 9 to:

a) 35 b) 78 c) 333 d) 1009 e) 1237 f) 3007 g) 2304 h) 999 i) 54843

j) Subtract 9 from the above.

Exercise 2

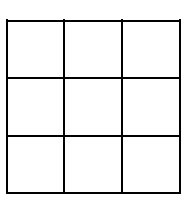
Solve the following:

a) 44 + 19 b) 44 – 19 c) 83 + 19 d) 83 – 19 e) 33 + 29 f) 77 – 29

g) 45 + 39 h) 88 – 49 i) 75 + 59j) 333 + 99 k) 444 – 99 l) 267 – 199

m) 487 + 299 n) 44 + 28 o) 83 – 38 p) 555 + 398 q) 465 – 297 r) 434 + 394

7) The 9-Squares Puzzle



You have the 9 numbers (1,2,3,4,5,6,7,8,9) and one number must go in each box above in such a way that each row and each column and both diagonals add up to the same number.

8) The 9-Dots Puzzle

♦ ♦ ♦

♦ ♦ ♦

♦ ♦ ♦

a) Can you join the 9 dots together using only four straight lines, without taking your pencil off the paper?

b) Can you do it with 3 lines only?

The 9-Nines Puzzle

You have 9 nines (9,9,9,9,9,9,9,9,9) and you have as many + and - symbols as you like. What is the largest total you can make with these, using at least one of these symbols?
For example, can you get a bigger number than the answer to

999 + 9999 + 99?

And what is the smallest total you can get with these same symbols?

Exercise:

At this point ask your teacher for more exercises.

Try to do this exercises mentally, using paper as little as possible.

7 Numbers with Shapes

Every number is a different expression of unity: each has its own unique qualities. All numbers fall into different categories, like odd and even numbers.

In this chapter we look at the structure of numbers and different classifications of numbers.

> **A factor** is a number that divides *exactly* into another number without a remainder.

For example 5 is a factor of 15, because 5 divides exactly into 15 without a remainder ($5 \times 3 = 15$).

However 6 is not a factor of 15, because 6 does not divide exactly. (There are two 6's in 15 and a remainder of 3).

Often we know the factors of a number by knowing our multiplication tables and working backwards. In the example above we know our 5 multiplication table and that 5 3's equal15, so both 5 and 3 are factors of 15.

1) Copy and complete the following table up to the number 30.

Number	Factors	Number of Factors
1	1	1
2	1,2	2
3	1,3	2
4	1,2,4	3
5	1,5	2
6	1,2,3,6	4
7		
8		

2) Which number in your table has the fewest factors?

3) How many factors does that number have?

4) Which numbers have the most factors?

5) How many factors do they have?

We see how different numbers are: 12, for example, has many factors, whereas 13 which is only one more than 12 has only two factors.

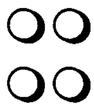
Square Numbers

You will need some counters. Use square paper.

• Start with one counter on the table:

1

• Add three more to form a larger square:



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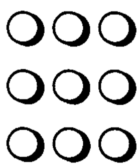
• Add five more to form a larger square:



3

6) Copy and complete the table below:

Number of Square	1	2	3	4	5	6
Number of Counters	1	4	9			

How many counters are in each square?

7) How many counters would you need for the 7th square?

8) How many counters would you need for the 11 th square?

9) How many counters would you need for the 30th square?

10) Describe how the number of counters increases as the square size increases.

11) Do you see a connection between each square number and the number of counters needed to make it?

> The numbers you have found; 1,4,9,16 and so on are called square numbers.

This is because you can arrange that number of counters to form a square. The 4 counters are in 2 rows of 2. The 9 counters are in 3 rows of 3 and so on.

We can also think of these numbers as being the answers to the following sums:

$1 \times 1 = 1$ $2 \times 2 = 4$ $3 \times 3 = 9$ $4 \times 4 = 16$ $5 \times 5 = 25$ etc.

So, if we 'square a number' we multiply it by itself. 2 squared is 4; 3 squared is 9; 4 squared is 16 and so on. We can also write square number as 2^2, 3^2 etc. 2^2 means 2×2, $3^2 = 3 \times 3$ and so on. In Sanskrit this is called *vargam* (squaring).

• Look at your table of factors completed earlier.

• Study your answers carefully to see what you notice. In particular, look at the square numbers and the number of factors column.

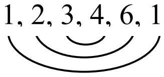
You may have noticed that square numbers all have an odd number of factors. 1 has only 1 factor; 4, 9 and 25 have 3 factors and 16 has 5 factors.

Square numbers always have an odd number of factors.
All other numbers have an even number of factors.

Factor Pairs

• Look at your table of factors. You will see that 12 has 6 factors: 1, 2, 3, 4, 6, 12. If we pair them up like this:

1, 2, 3, 4, 6, 12



do you notice anything?

If you multiply the numbers in a pair you get 12:

$1 \times 12 = 12$,
$2 \times 6 = 12$,
$3 \times 4 = 12$.

These pairs of numbers are known as factor pairs.

This pairing is useful, because if you know one factor of a number you can get another. For example, if you know 4 divides into 44 then another factor must be 11 because $4 \times 11 = 44$.

12) Write out the factors of 20 and show how they pair up.

13) Do this also for 18, 22, 24, 29.

14) For each of the following numbers, write down two pairs of factors: 32, 35, 40, 42, 45, 48, 55.

Factor Rectangles

We can also represent factor pairs by rectangles. For example, the factor pairs of 12 can be represented in the following way:

OOOOOOOOOOOO **1×12=12**	OOOOOO OOOOOO **2×6=12**	OOOO OOOO OOOO **3×4=12**

• Try making these with your counters and then draw them in your book.

15) Draw as many rectangles as you can for the number 20.

16) Draw rectangles also for 18, 22, 24, 7, 17, 16.

Since the factors of 16 are 1, 2, 4, 8, 16 we can pair up the 1 and 16, and the 2 and 8.
But this leaves the 4. Clearly 4 × 4 = 16 and so we need to pair 4 with itself.

So you should have drawn three rectangles for the number 16 (A square is a special kind of rectangle).

This will apply to all square numbers since they always have an odd number of factors.

Prime Numbers (abhedya)

You may have noticed that some of the numbers you were drawing rectangles for could only be drawn in one way: a single row of counters. The number 7, for example, can only be drawn as a row of 7 counters. It has only factors of 1 and 7.

> Numbers which have exactly two different factors are known as **prime numbers**.
> Their only factors are the number itself and one.

As we will see later prime numbers are very important.

The number 1 is not normally considered as being prime because according to the above definition, prime numbers have exactly two different factors.

• Check that 2, 3, 5, 7, 11, 13,17 are all prime numbers.

17) Write these numbers out and add all the other prime numbers up to 30 to the list.

18) Are there any even numbers in your list that are prime numbers (apart from the number 2)? If yes, please explain. If not, why not?

19) Which of the following are prime numbers: 31, 32, 36, 37, 39.

The Sieve Of Eratosthenes

The Greek mathematician Eratosthenes found a neat way of deleting all non-prime numbers so that only the primes were left.

• Write out the numbers from 1 to 100: write 1 to 10 on the first line, then 11 to 20 underneath these on the next line and so on until you reach 100.

• The first prime number is 2, so cross out every 2nd number after 2 (4, 6, 8 etc.).

• The next prime number is 3, so cross out every 3rd number after 3 (6, 9, 12 etc.).

• The next prime number is 5, so cross out every number after 5.

• Similarly cross out every 7th number after 7.

The next prime number is 11 but there is no need to go this far as no new numbers will be crossed out: the numbers not crossed out are the prime numbers from 1 to 100.

Triangular Numbers

You will need some counters. Again, use square paper.

• Start with one counter on the table:

❍ **1**

• Add two more to form a triangle:

❍
❍ ❍ **2**

• Add three more to form a larger triangle:

❍
❍ ❍
❍ ❍ ❍ **3**

20) Copy and complete the table below:

Number of Triangle	1	2	3	4	5	6	7	8
Number of Counters	1	3	6					

How many triangles can you find in each triangle?

21) How many counters would you need for the 9th triangle?

22) How many counters would you need for the 30th triangle?

23) Describe how the number of counters increases as the triangle size increases.

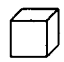
24) Do you see a connection between each triangle number and the number of counters needed to make it?

25) Write down 2 triangle numbers which are also square numbers.

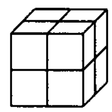
Cube Numbers

You will need some multilink cubes.

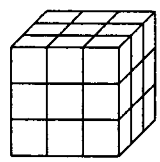
• Start with one cube on the table:


1

• Add 7 more to make a larger cube:


2

• Add 19 more to make a larger cube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26) Copy and complete the table below:

Number of Cube	1	2	3	4	5	6
Number of cubes	1	8	27			

27) How many cubes would you need for the 7th cube?

28) How many cubes would you need for the 10th cube?

29) How many cubes would you need for the 30th cube?

30) Describe the connection between the cube number and the number of cubes it is made from.

31) Which of your cube numbers are also square numbers?

The numbers you have found are called **cube numbers**.
This is because you can arrange that many cubes to form a large Cube.
The length, breadth and height of cubes are always the same.

We can also think of these numbers as being the answers to the following sums:

$1 \times 1 \times 1 = 1$
$2 \times 2 \times 2 = 8$
$3 \times 3 \times 3 = 27$
$4 \times 4 \times 4 = 64$ etc.

Cubing in Sanskṛit is called *ghanam*.

So, if we cube a number, we multiply it by itself twice. 5 cubed is $5 \times 5 \times 5 = 125$. We can also write cubed number as 5^3. 5^3 means $5 \times 5 \times 5$.

This is similar to squaring a number where a number is multiplied by itself: 5^2 means 5×5.

32) What does 6^3 mean?
33) What does this equal?

34) What do you think 10^3 means?
35) What does this equal?

36) What do you think 6^4 means? (we say '6 to the power of 4')
37) What about 7^6 ? (we say '7 to the power of 6')

Summary of Number Sequences

The various sequences studied in this chapter will be referred to and used in future work; thus, they are listed here for easy reference.

In each case the sequence goes on forever.
Square numbers: 1, 4, 9,16, 25, 36 ....
Prime Numbers: 2, 3, 5, 7,11,13 ....
Triangle Numbers: 1, 3, 6,10 ....
Cube Numbers: 1, 8, 27, 64 ....

Vedic square and its patterns

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

Calculation of the Vedic Square

1	2	3	4	5	6	7	8	9
2	4	6	8	1	3	5	7	9
3	6	9	3	6	9	3	6	9
4	8	3	7	2	6	1	5	9
5	1	6	2	7	3	8	4	9
6	3	9	6	3	9	6	3	9
7	5	3	1	8	6	4	2	9
8	7	6	5	4	3	2	1	9
9	9	9	9	9	9	9	9	9

Vedic Square

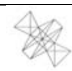
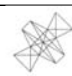
The Vedic Square is a square of nine tables of multiples of the basic nine numbers. To make a Vedic Square, we must have a square with nine equal divisions on each side By joining the dividing marks on all sides, we get a square with 81 units of equal size. Now write out numbers 1 through 9 across the top row and down the left column. All remaining rows are created by multiplying the elements in these two rows. All double numbers are reduced (by addition) to single whole numbers. When you put the numbers under the table of 2 and 3 for example, you have to use the following method:

$2 \times 1 = 2$	$3 \times 1 = 3$	$4 \times 1 = 4$
$2 \times 2 = 4$	$3 \times 2 = 6$	etc.
$2 \times 3 = 6$	$3 \times 3 = 9$	
$2 \times 4 = 8$	$3 \times 4 = 12 \ \ 3$	
$2 \times 5 = 10 \Rightarrow 1$	$3 \times 5 = 15 \Rightarrow 6$	
$2 \times 6 = 12 \Rightarrow 3$	$3 \times 6 = 18 \Rightarrow 9$	
$2 \times 7 = 14 \Rightarrow 5$	$3 \times 7 = 21 \Rightarrow 3$	
$2 \times 8 = 16 \Rightarrow 7$	$3 \times 8 = 24 \Rightarrow 6$	
$2 \times 9 = 18 \Rightarrow 9$	$3 \times 9 = 27 \Rightarrow 9$	

Play Of Opposites

The Vedic Square clearly reveals the play of opposites that occurs in the process of multiplication . The visual pattern created by combining the number 1 to the number 4 forms a diagonal, connecting the top left and bottom right portions of the square. And the pattern created from linking number 5 to number 8 is just the opposite, connecting the top right corner with the bottom left portion of the square. Patterns 1 and 8 are exact opposites of each other, as are 2 & 7, 3 & 6, and 4 & 5. Nine makes its own unique pattern, not complemented by any other number in the Vedic Square.

These "opposites of nature" can also be found in the numerological progressions. In column 1 of the Vedic Square the numbers progress sequentially from 1 to 9, whereas in column 8, which is opposite of number 1, they go in reverse order. Similarly, this is the case with 2 & 7, 3 & 6, and 4 & 5.

Number	Ruling Planet	Geometrical shape	Basic attribute
1	Sun (Surya)		Light
2	Moon (Chandra)		Affection
3	Jupiter (Guru)		Guru of demigods
4	Rahu		Malefic
5	Mercury (Buddha)		Benefic
6	Venus (Śukra)		Guru of demons
7	Ketu		Detached
8	Saturn (Śani)		Darkness
9	Mars (Mangal)		Leadership

Zero

Zero itself has no value when it is alone, but in combination with other 9 numbers it gives a birth to arithmetical progressions and the series of double, triple and other multiple numbers. Without knowing about zero, we cannot count beyond nine, --beyond the material nature. In Sanskrit, zero is called *śūnya* or void and it indicates Brahman.

More on number nine

We already discussed number 9 in chapter two. Here are a few qualities that we didn't cover yet.

Number 9 is beyond the eightfold *prakriti*, which consists of 3 gunas and 5 elements. 9 is the number of consciousness. *Prakriti* (consists of numbers1-8) and consciousness (represented by the number 9) creates the manifested world of names and forms. From the Vedic Square, we see that numbers 1-8 gradually decrease, but when they come in contact with 9 they immediately gain their full power and become 9.

Number 9 is ruled by the planet Mars, whose Lord is Narasimhadeva. That is why 9 is so special.

Exercise 1
Ask your teacher for more exercises.

8 Geometry (rekhāganitam)

Geometry is the third of the three branches of mathematics.
Geometry deals with forms: lines, circles, solid objects, and the ways in which they are structured and compared.

All forms exist in space and all geometrical objects have some kind of space, whether it is length or space inside it or even just its position near some other object.

The simplest geometrical form is the point (*bindu*). All other forms are composed of points and so, in one sense, the point contains all geometrical forms.

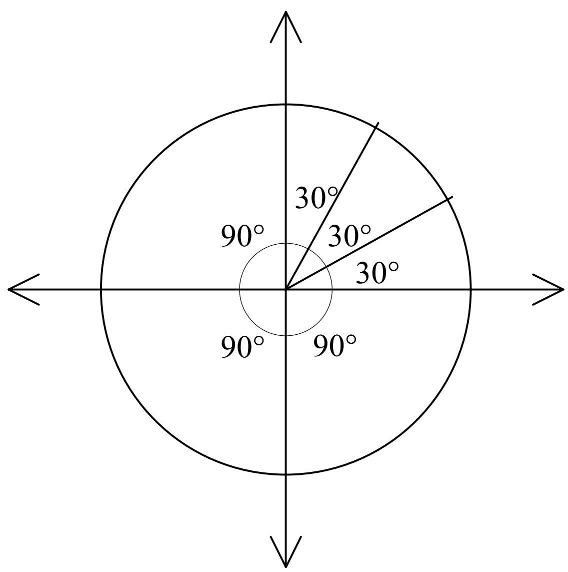
By moving, a point can create a straight line (*rekhā*) or a circle; circles can create a spheres and lines create planes (*talam*), which determine space (*kham*).
Complex shapes can be made from simple shapes.

Definitions of terms and shapes

Angle (*koṇa*)

> A point on a plane bears angle (*koṇa*). It is divided into 12 *rāśi*, each of which is divided into 30 *aṁśa* (degrees). Each *aṁśa* is divided into 60 *kalā* (minutes) and each *kalā* is divided into 60 *vikalā* (seconds). Altogether there are $360°$ (*amśa*) around a point.

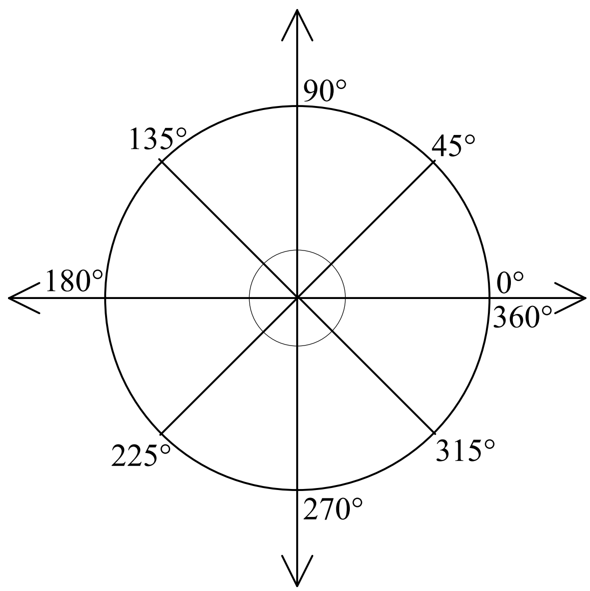
One fourth of the total *koṇa* ($90°$) is known as *lambakoṇa* (right angle). Angle less then $90°$ is called *nyūnakoṇa* (acute angle) and angle between 90 and 180 *amśa* is called *bṛhatkona* (obtuse angle).

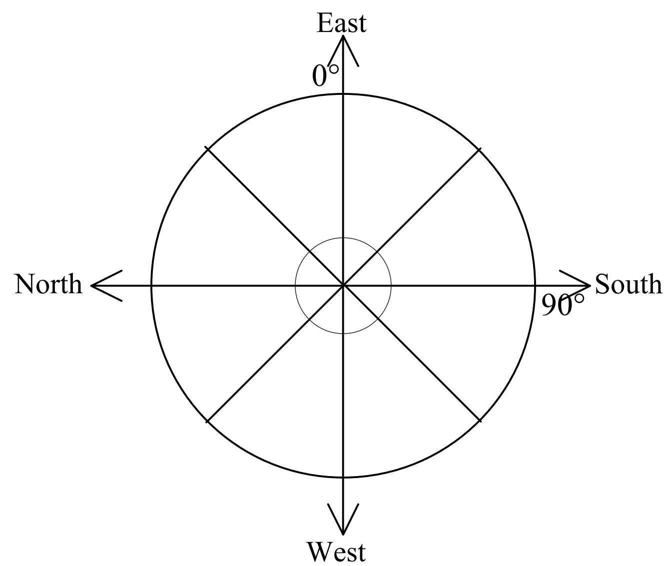


Here we have a circle. As we said, there are $360°$ around it. So if we put $0°$ at our right and go counter clock wise (CCW) then the number of degrees will increase. When we approach the $90°$ angle to $0°$, we see that it is right angle. When we continue we come to the point of $180°$, which is $2 \times 90°$ and it is opposite to $0°$. Then we come to $270°$, which is three times right angle. Finally we come to $360°$ which at the same point as $0°$. That is how we count angles in geometry.

For determining directions, we count clock wise (CW)! So if we consider East at $0°$, then can you say at what degree will be South, West and North?

According to the standards established recently, the world follows that North is at the $0°$.

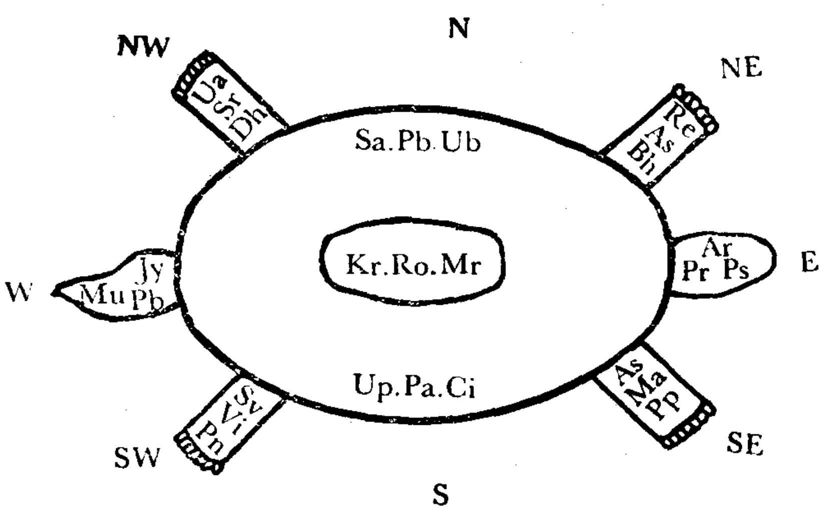


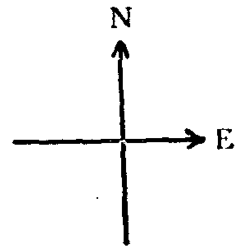


Kūrmavibhāga (Division Of The Globe)

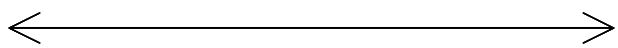
The following is definition of direction according to Nārada Purāṇa. For now, just read it; in later chapters we will come back to it.

The land of the world has been laid out as if divided into nine sections *(mandala)* in a clockwise direction on nine parts of *kūrma* (a tortoise) facing East. The parts are placed according to the directions, i.e. E, SE, S, SW, W, NW, N, NE and centre corresponding to head, right foreleg, right flank region, right foot, tail, left foot, left flank region, left foreleg and navel respectively. Each of the regions consists of different parts of the world's land and at the same time it corresponds to respective parts of our bodies. The land of the world is ruled by different *nakṣatra* (stars) which we will discuss in later chapters.

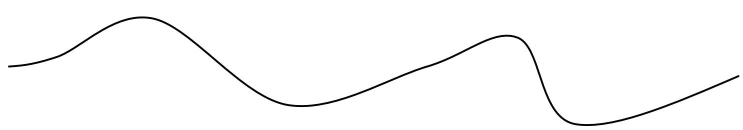




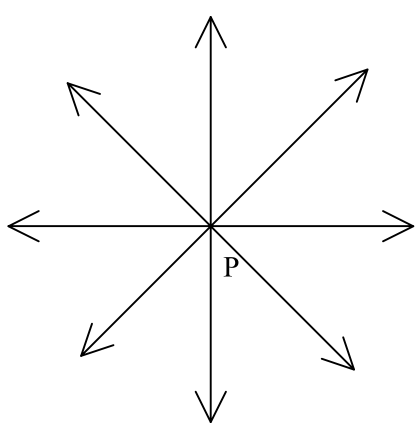
Straight line (*ṛijurekhā*)

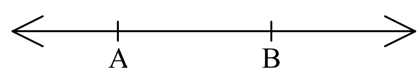


Curved line (*vakrarekhā*)



An infinite number of lines can pass trough a point, but only one line can join two different points.





A section of a straight line with two end points (*agrabindu*) is called a line segment (*khaṇḍarekhā*).

Circle (*maṇḍalam*) or (*vṛttam*)

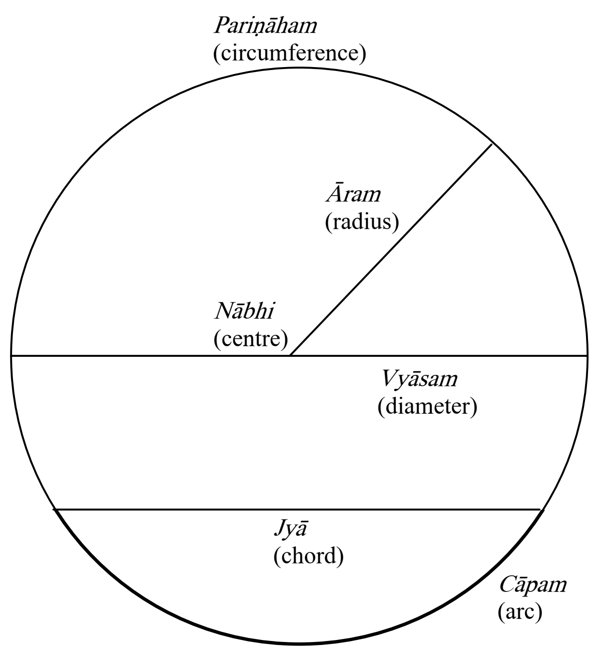
The simplest geometrical figure is a circle (*maṇḍalam* or *vṛttam*). It was drawn using a thread (*śulbam*) tied to a *śanku* peg. The fixed central point is known as *nābhi* and the path of the moving point is called *pariṇāham* or circumference. The distance from the centre to the cirumfrence is known as *āram* or radius and double the radius is known as *vyāsam* or *viṣkambham* or diameter. The ratio of circumference to diameter, now referred to as π was originally named *paridhi*.

$$\pi = \frac{\text{circumferrence}}{\text{diameter}} = \frac{\text{pariṇāham}}{\text{vyāsam}}$$

or

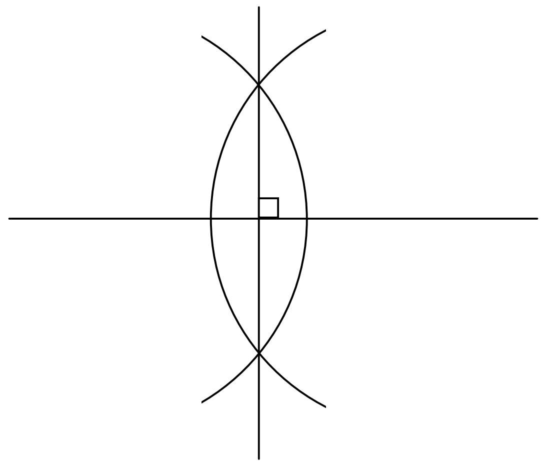
$$\text{pariṇāham} = \pi \text{d} = 2\pi \text{r}$$

A portion of a circle is known as *cāpam* or arc. A line joining two points of a circle is *jyā* or chord.

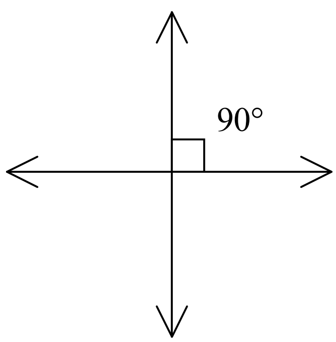


Construction of perpendicular - *lambāmiti*

Draw a line segment – base line. Draw arcs from both end points so as to cut two points. This looks like a *matsya* with one and as *mukham* (face) and the other end as *puccham* (tail). The line joining the face and tail will be perpendicular to the base line. This process is known as *matsya-mukha-pucchayoryojanaṁ lambāmiti*.



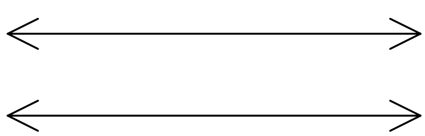
Perpendicular lines (*lambarekhā*) cross each other at right angles. We call it a right angle because a square fits into it without leaving any gaps.



The walls of a room are built at right-angles to the floor.
A page of a book usually has a right angle in each corner. The plus sign + has four right angles.

Another way of saying that two lines are at right-angles to each other is to say that they are perpendicular. For example, you could say that the wall is perpendicular to the floor.

Parallel lines (*samāntararekhā*) run next to each other and they will not cross even if you extend them as long as you like.



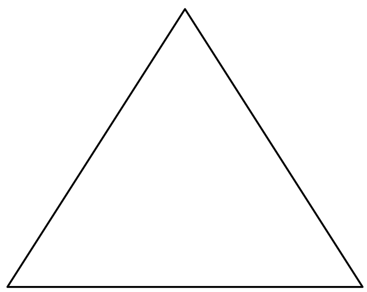
- List at least three other places where you can see right angles.

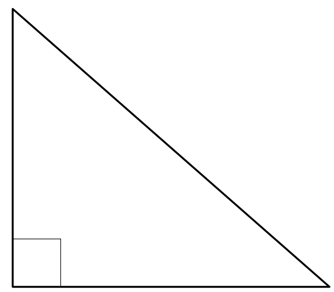
- Now make yourself a rectangle out of thin card or paper. The size is not important, but make sure that the corners are as near to perfect right angles as you can make them. You may find it helpful to use a set-square to make it.

• Can you think of a way to make a square of a long piece of cloth? How can you measure a *chaddar*, which is generally twice as long as it is wide.

1) Can you think of a mathematical symbol that is a pair of parallel lines?

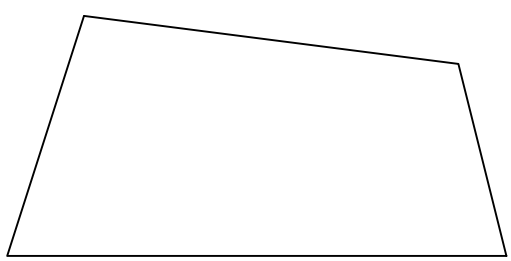
2) Name a mathematical symbol that is two perpendicular lines.

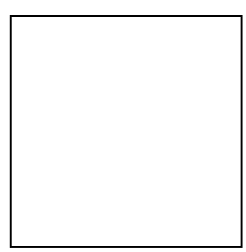


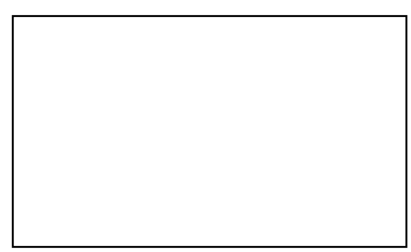


A triangle (*tryaśram*) or (*sāmānya tribhujam*) is a geometrical figure consisting of three sides. The total sum of the internal angles is $180°$.

A triangle in which one angle is right is called *lambatribhujam* or *ardha-chaturasrakam* (right angle triangle)
A triangle with all three sides same is called *sama-tribhujam* (equilateral triangle).
A triangle with just two sides equal is called *dvi-sama-tribhujam* (isosceles triangle).
A triangle with no sides equal is called *viṣama-tribhujam* (scalene triangle).





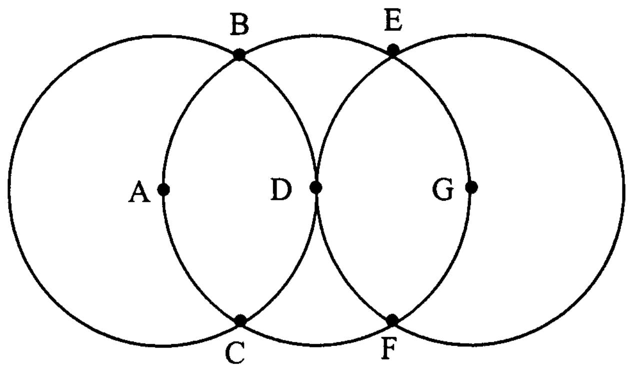


A geometrical figure that is formed by four sides is called *caturbhujam* or quadrilateral. It has four angles and the sum of those angles is $360°$.

Quadrilateral figures, in which all angles are right but the sides next to each other are not equal, are called rectangles (*dīrghacaturaśram*).

In a rectangle, if all the sides are equal then it is called a square (*samacaturaśram*).

In the diagram below, we have three circles. Some points have been labeled. The distance between D and B is 3 cm.



A is the centre of the circle on the left.
The circle on the left and the middle circle meet at B and also at C.

Exercise 1

Copy and fill in:

a) The centre of the middle circle is at __.
b) The centre of the circle on the right is at __.
c) The middle and right circles meet at __ and at __.
d) The left and right circles meet at __.
e) The length of the radius of each circle is __ cm.
f) The distance from A to G is __ cm. (We can write this as AG = 6cm)
g) BE = __ cm.
h) BD = __ cm.
i) BG = __ cm.

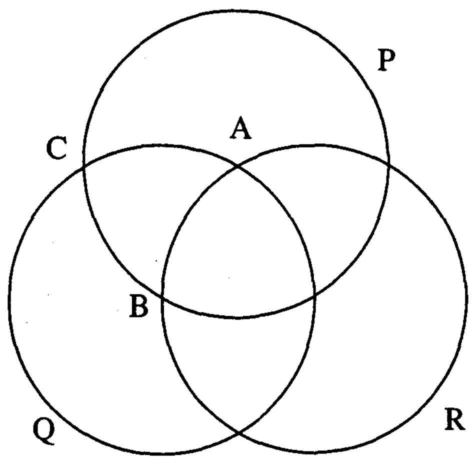
• Now copy the diagram into your book exactly, using compasses.

• With your ruler and a colored pencil join C to F, join F to E and join E to C.

The triangle CFE is a **right-angled triangle** because a right angle would fit into the triangle at F.

ABD is equilateral triangle because all its sides are equal in length.

Diagram below shows three circles P, Q and R.



• Copy the diagram into your book. Copy all the letters as well.

• There are six points where the circles meet. Count them.

• Circles P and Q meet at C and D. Put D in the correct place on your diagram.

• Circles P and R meet at B and F. Mark F on your diagram.

• Q and R meet at E and A. Put E on your diagram.

3) Join points A, B and D in color to make a triangle.

4) There are other equilateral triangles that can be drawn in the diagram. For each one you can find write down the three letters that describe it.

Right-Angled And Equilateral Triangles

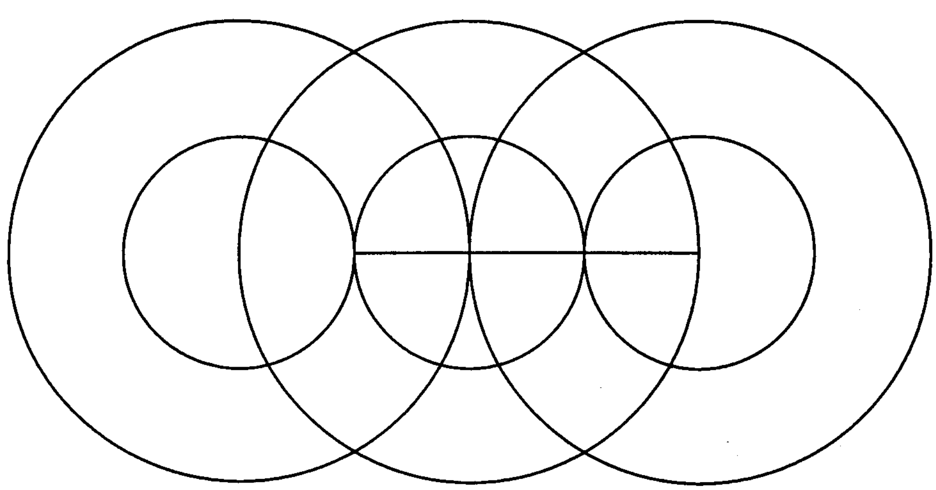


Diagram D

In the diagram above there are 3 small circles each with a of radius l.5cm.

• Copy these 3 circles exactly.

In the diagram there are also 3 large circles. Each of these has the same centre as the small circles and is twice the radius of the small circles,

• Copy the 3 large circles onto your diagram.

• Draw in the straight line as well. Measure the length of this line. It should be 4.5cm.

5) Now draw a right-angled triangle with a base of 4.5cm on your diagram in color.

6) Draw an equilateral triangle with sides 3cm on your diagram.

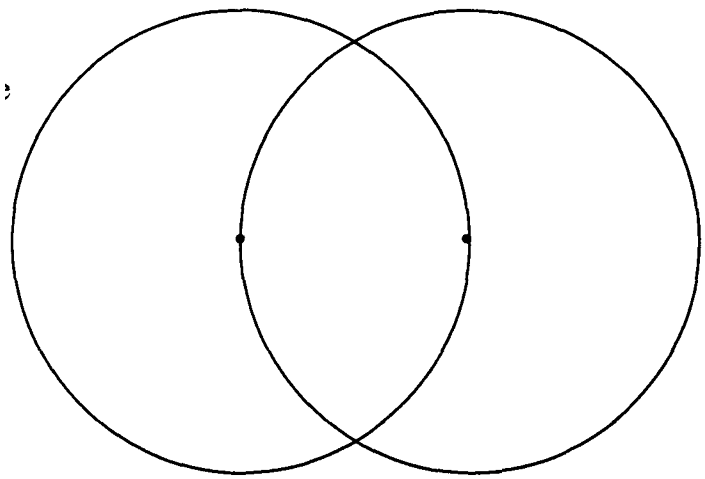
Exercise
Ask your teacher for more exercises.

9 Symmetry

Many forms in nature are symmetrical like flowers, the faces of people, the sun, the moon, and so on. There are two main types of symmetry: line symmetry and point symmetry (also called rotational symmetry). We will look first at line symmetry.

A)

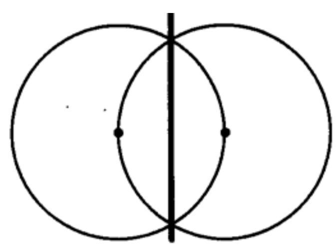
● Draw the diagram opposite using compasses. A black dot is at the centre of each circle. (the radius is 3cm)



When lines meet we say that they **intersect** and the point where they intersect is called the **point of intersection**.

● On your diagram put a dot at each of the two intersection points.

● Join your dots up like this:



This line is called a line of symmetry because if you fold the paper along this line one half would fold over onto the other half The pattern is symmetrical.

If you stand a mirror along the line you will find that one half of the pattern is a reflection of the other.

Your diagram has another line of symmetry.
● Can you see it? It goes through the first pair of dots.

1) Draw the second line of symmetry in.

There are no other lines of symmetry.

B)

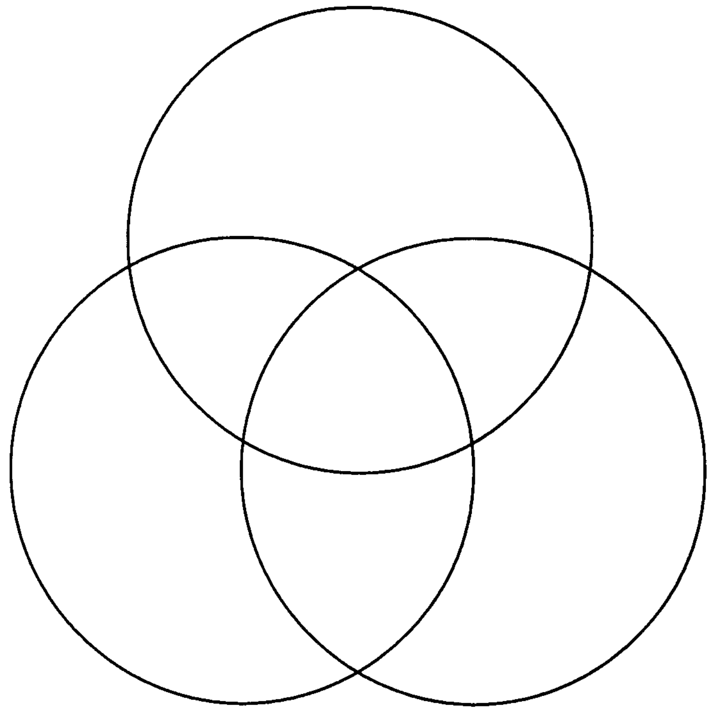
● Draw the two circles again but without any dots or straight lines.

● Place your compass point at the upper intersection point and draw another circle the same size as the others.

● Where do you think the centre of this diagram is?

● Put a yellow dot where you think the centre is.

This pattern has 3 lines of symmetry. Draw the 3 lines in.



C)

● Now draw the 3 circles again, but with a radius of 2cm. Leave some space around them.

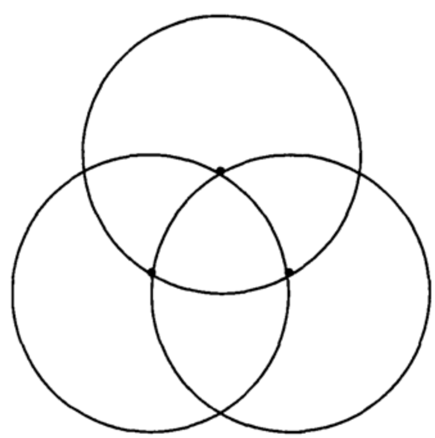
● Put a red dot at the centre of each of the 3 circles.

There are 3 other intersection points.

● Can you see them?

● Draw a blue dot at each of these 3 points.

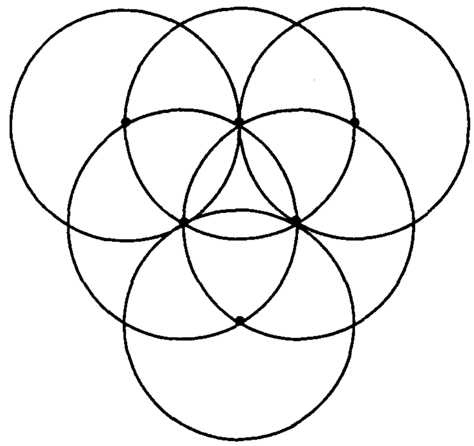
● Draw 3 more circles the same size as before, centered on the blue dots.



How many new intersection points have appeared?
There should be 6.

● Put a red dot at each of these points.

● Draw a circle centered on each of these red points.



You should have 9 new intersection points now.

● Put a blue dot on these intersection points.

● Using a ruler, join these up in order with a colored pencil.

A 9-sided figure is called **nonagon**.

● Write this word beside your diagram.

3) How many lines of symmetry does this nonagon have?

4) Draw them in.

Polygons (*bahu-bhujam*)

You will need worksheet 1, which is an extended version of the pattern you have been drawing. Write the title **Polygons** at the top.

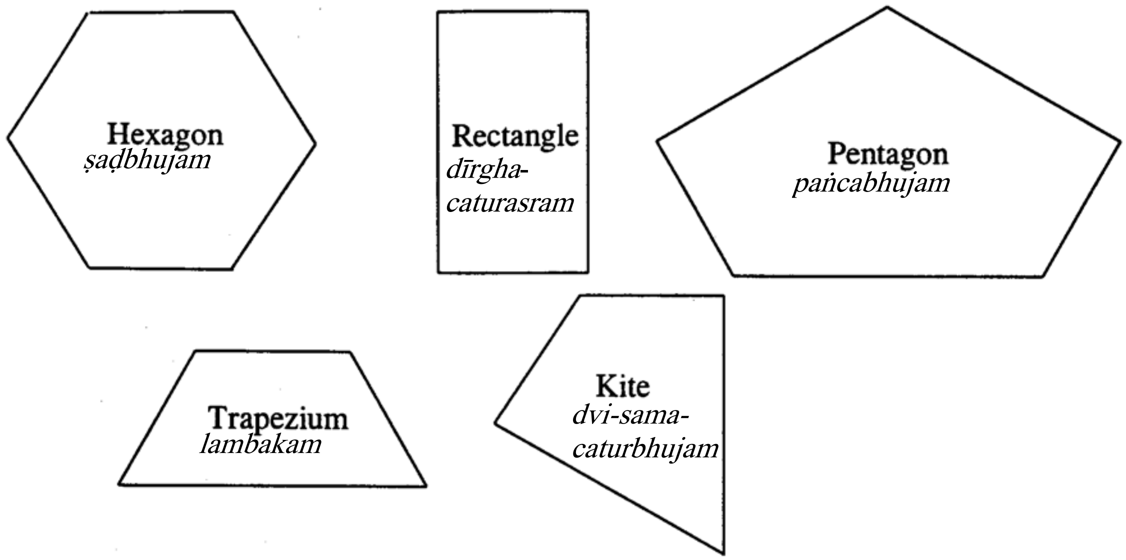
You will be joining up intersection points in the pattern on the worksheet to make shapes. You will need a ruler and some colored pencils.

Can you see which intersection points to join up on the Worksheet to make this shape?

● Join the points up using a colored pencil.

In visualizing a shape not actually there you are using the formula called, “By Mere Observation.”

5) Find the shapes below and join them up. Write the name of the shape as well.



A 3-sided shape like the one you have drawn is called a triangle.
This particular triangle, you may remember, is an equilateral triangle (all sides are equal).

A 4-sided shape is called a quadrilateral.

6) 3 of the shapes you have drawn are quadrilaterals. Which 3?

A 5-sided shape is called a pentagon.

A 6-sided shape is called a hexagon.

The particular hexagon you have drawn would be called a regular hexagon. The word regular tells you that all the sides are equal in length.

7) Is the pentagon you drew a regular pentagon?

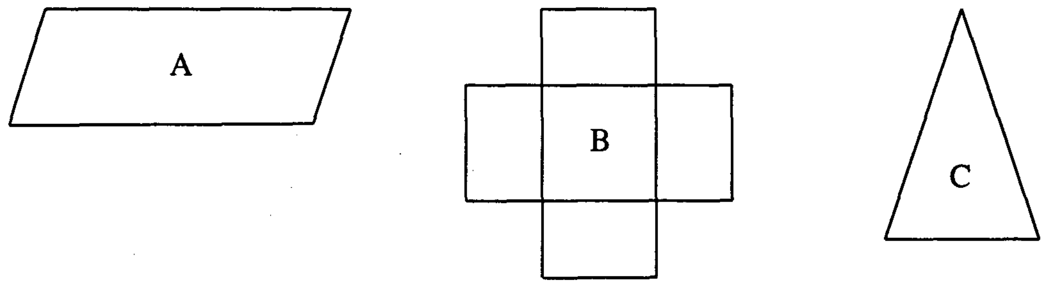
The shapes you have drawn are different types of polygons: A polygon is a many-sided figure.

- Copy into your book:- A triangle has 3 sides.

A quadrilateral has 4 sides.
A pentagon has 5 sides.
A hexagon has 6 sides.
A regular polygon has all its sides of equal length.

Rotational Symmetry

Look at the three shapes below.



Shape A does not have any lines of symmetry, but it does have a kind of symmetry.

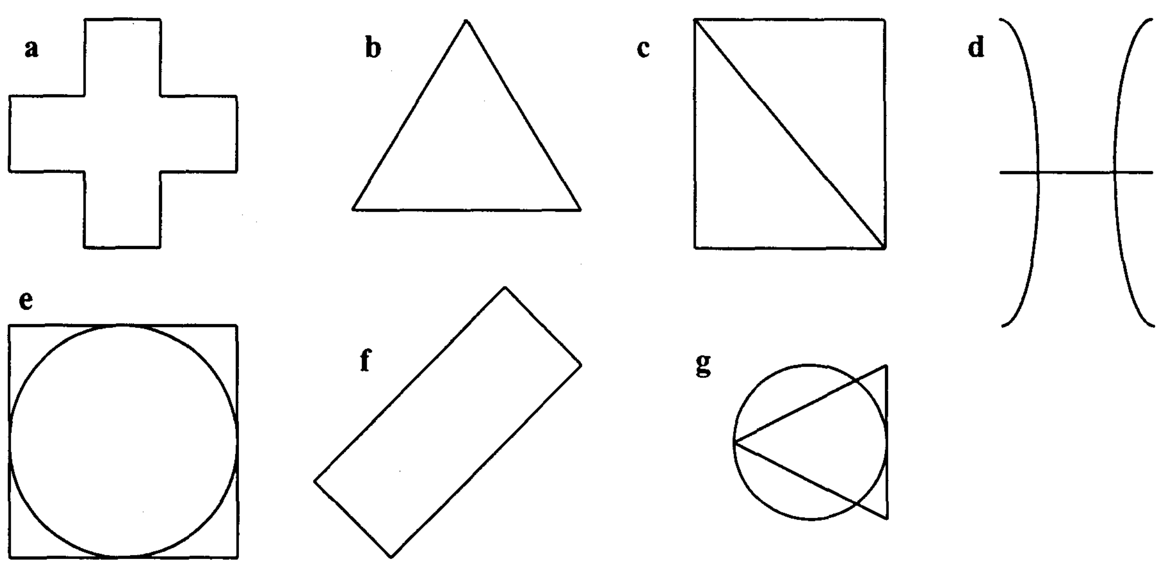
If this shape was rotated half a turn about its centre, it would coincide with its original position.
So 2 half turns would bring it back to its original position.
We say that the shape has rotational symmetry of order 2.

Similarly, if shape B was given a quarter turn it would coincide with its original position and 4 quarter turns would return the shape to its original position/ It has rotational symmetry of order 4.

Shape C requires 1 complete turn before returning to its original position, so it has rotational symmetry of order 1.

Exercise 1

Copy the following shapes, put a dot at the centre of each shape (if it has a centre) and write down the order of its rotational symmetry:



Exercise 2

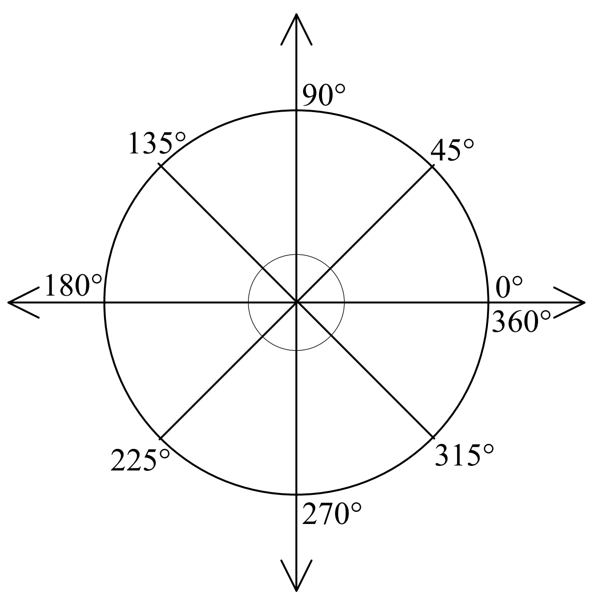
Ask your teacher for more exercises.

10 Angles and triangles

An angle measures the amount of a turn.

A complete turn is usually divided into 360 degrees, written $360°$.
This means that half turn is $180°$.
A quarter turn, which is a right angle, is $90°$.

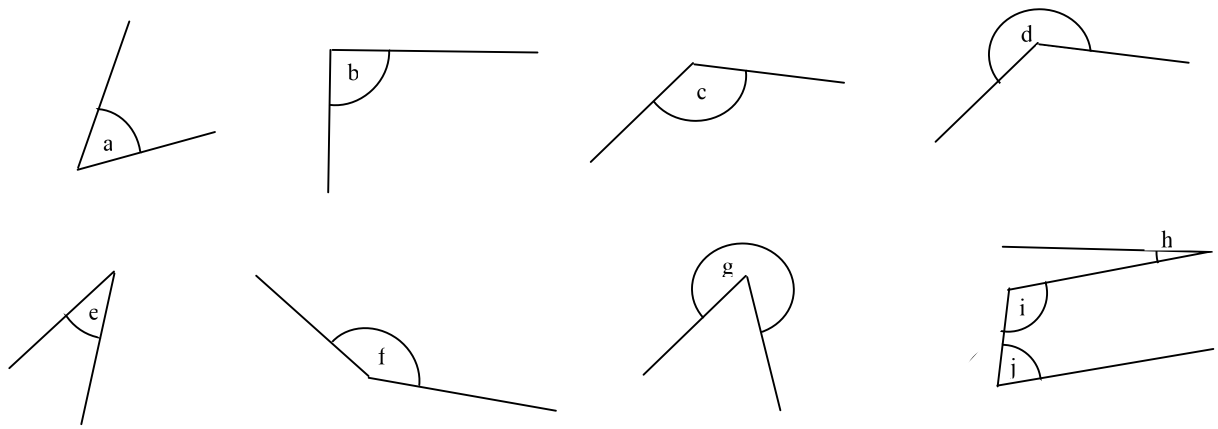
All *tīkṣṇa-aṁśa* (acute angles) are those between 0 and 90 degrees.
Angle 90 degrees is called *lamba-aṁśa* (right angle).
All *vitīkṣṇa-aṁśa* (obtuse angles) are between 90 and 180 degrees.
180 degrees is *ardha-pūrṇa-aṁśa*
All *apatīkṣṇa-aṁśa* (reflex angles) are between 180 and 360 degrees
270 degrees is called *tri-pada-pūrṇa-aṁśa*.
360 degrees is called *pūrṇa-aṁśa* (complete angle).



Exercise 1

Draw the diagram into your exercise book.

- With a colored pencil highlight a range in which acute angles are found.
- Find angle $120°$. What is a name of such an angle?
- Where is $260°$?



a) Which of the angles above are acute?

b) Which are obtuse angles?

c) Which are reflex angles?

Exercise 2

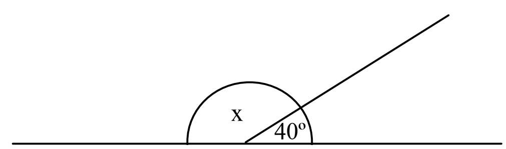
Now use the angles given below and sketch the angle approximately (use pencil and ruler).

a) 20º b) 100º c) 160º d)70º e)200º f) 260º g)300º h) 45º

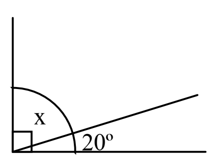
Finding Angles

Example 1

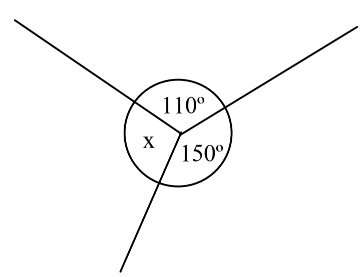
Knowing that a $360°$ makes a full turn, we can find unknown angles.
Find the angles marked x:



a) Since the two angles together make 180º and one of them is 40º the other must be 140º; x = 140º.



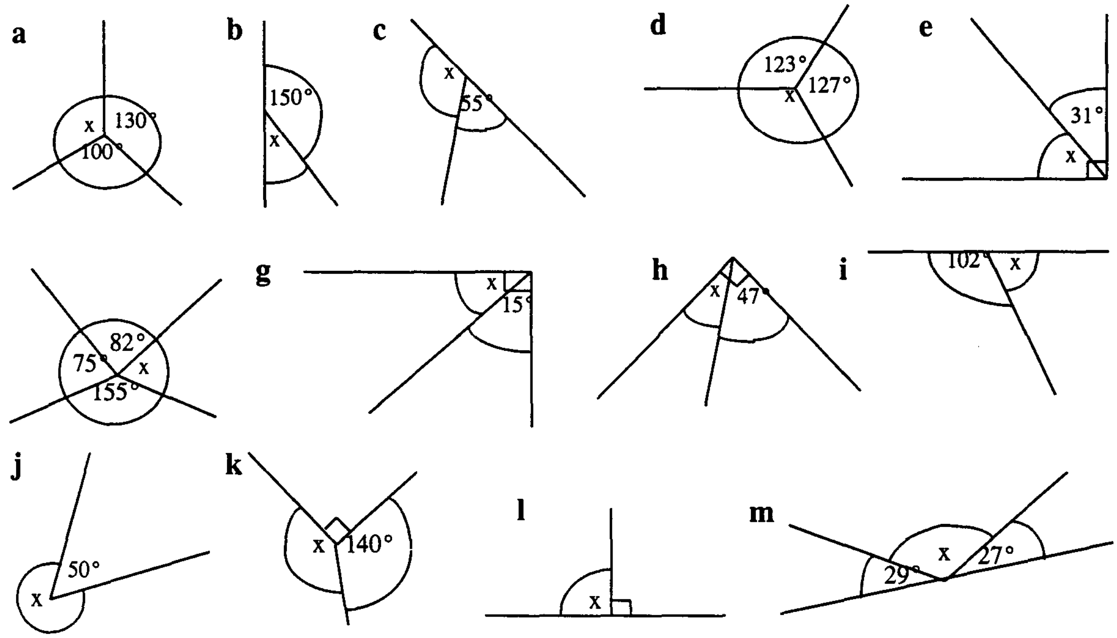
b) The square indicates a right angle, which is 90º, so x = 70º.



c) The three angles make 360º so "By Addition and By Subtraction" we get x = 100º.

Exercise 3

Find x:

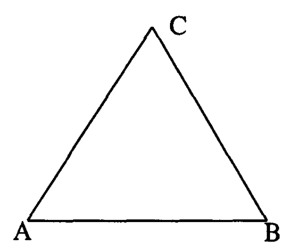


Triangles

A triangle (*tryaśram*) or (*sāmānya tribhujam*) is a geometrical figure consisting of three sides. The total sum of the internal angles is 180°.

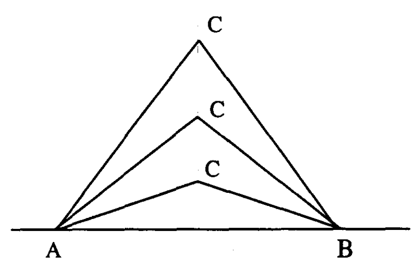
A triangle in which one angle is right is called *lambatribhujam* or *ardha-chaturasrakam* (right angle triangle)
A triangle with all three sides same is called *sama-tribhujam* (equilateral triangle).
A triangle with just two sides equal is called *dvi-sama-tribhujam* (isosceles triangle).
A triangle with no sides equal is called *viṣama-tribhujam* (scalene triangle).

Triangles are 3-sided figures that are extremely useful in mathematics. The simplest triangle is the equilateral triangle which has three sides that are equal in length. This triangle has 3 lines of symmetry.



4) Copy the triangle and draw in the 3 lines of symmetry. Write “equilateral triangle” beside it.

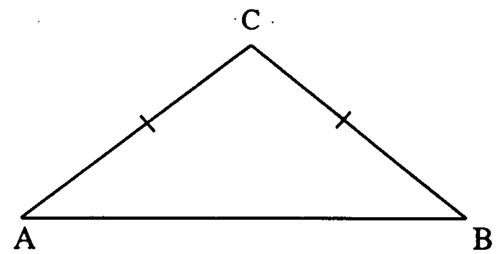
Now suppose that the base of the triangle is fixed but the corner C is allowed to move up or down. Regardless of how high or low Angle C is placed, would you agree that the lines AC and BC would be equal in length? (triangle with obtuse angle is called *vitīkṣṇa*.)



Angles And Triangles

Suppose the base of a triangle is 5cm wide and that C is 2cm above the base.

In that case, we would say that the **height** of the triangle is 2cm.



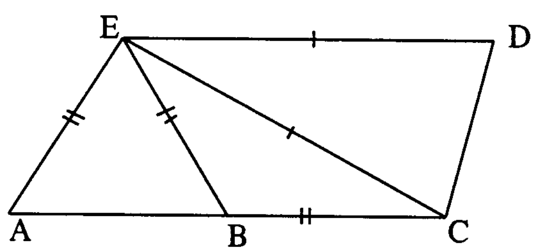
5) Draw this triangle and label it A, B and C.

A triangle like this which has just 2 equal sides is called an isosceles triangle.

- Write **isosceles triangle** beside your diagram.
- Draw in the line of symmetry.

You will see on the diagram shown that the equal sides are marked to indicate that they are equal. Place these marks on your triangle.

The equal sides will not always be marked with a single dash like this; there may be 2 or 3 marks. Sides with the same marking are equal.

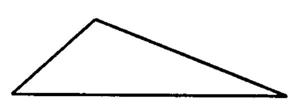


So in this diagram AE, BE and BC are al equal in length.
EC and ED are also equal to each other.

- How many isosceles triangles do you see in the above diagram?

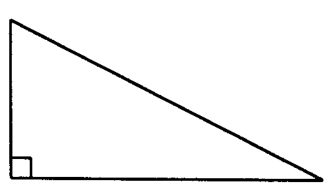
6) Draw another isosceles triangle with a base of 7cm and a height of 4 cm. Mark the equal sides.

If a triangle is not equilateral or isosceles it is called scalene. So a triangle in which no sides are equal is a **scalene triangle**.



Finally a triangle which has a right angle in it is called a **right-angled triangle**.

It is usual to put a small square in right angles. So the square tells you that the angle is $90°$.

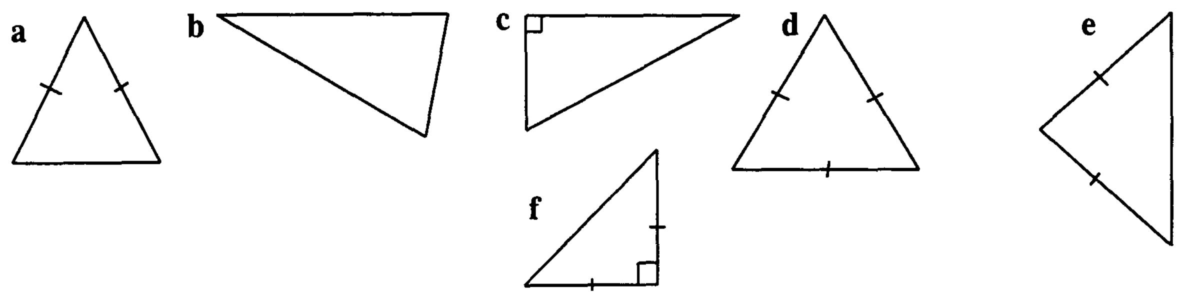


Exercise 4

For each triangle below say if it is

E – Equilateral
I – Isosceles
S – Scalene
R – Right-angled

(more then one will apply in some cases)

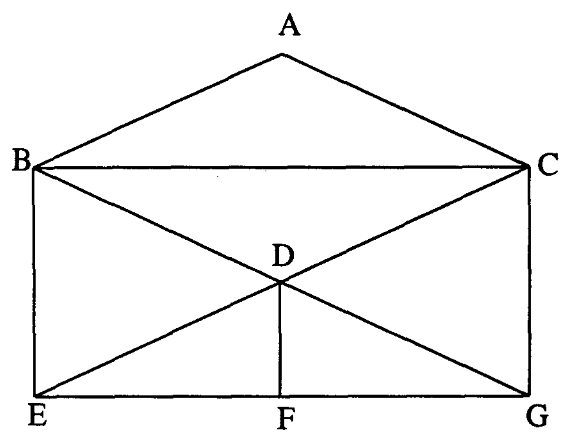


The diagram shows a figure ABEGC. It is described by its **vertices** (its corners) which are A,B,E,G and C.

You could say that it is composed of a rectangle BEGC and a triangle ABC.

Or you could say it is composed of 6 triangles.

7) Describe the 6 triangles in terms of vertices.
8) Which line is parallel to BC?
9) Which 2 lines are parallel to DF?
10) Which line is parallel to EC?



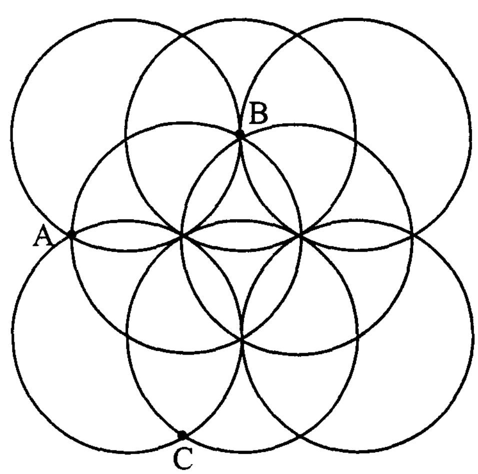
● Study this diagram.
● Measure the radius of a circle and draw the diagram carefully.
● Complete triangle ABC.

11) What kind of triangle is it?

12) Draw a line through C that runs parallel to AB.

13) Draw in the rectangle BACD.

14) Find the point E which is on the line AB, and BE= 1 cm? Mark the point E on your diagram.



Exercise 5
Ask your teacher for more exercises.

Magic Squares

A Magic Square is a square in which the numbers in every row, column, and diagonal adds up to the same total.

For example, this is a Magic Square because all 3 rows add up to 15, and so do each of the 3 columns and the 2 diagonals.
All the numbers from 1 to 9 are used.

8	**1**	**6**
3	**5**	**7**
4	**9**	**2**

● Check that you agree with this.

So there are 8 lines totaling 15.
In one of these lines the numbers increase by 1; this is the diagonal 4, 5, 6.

1) In which line do the numbers increase by 2, by 3, by 4?

Here is a 5 by 5 Magic Square (it has 5 rows and 5 columns).

23	**1**	**2**	**20**	**19**
22	**16**	**9**	**14**	**4**
5	**11**	**13**	**15**	**21**
8	**12**	**17**	**10**	**18**
7	**25**	**24**	**6**	**3**

2) Find out what the line total is and check that all 12 lines add up to this number.

In the centre of this 5 by 5 square is a 3 by 3 square.

3) Draw out this 3 by 3 square including the numbers in each box. What do you notice about this 3 by 3 square?

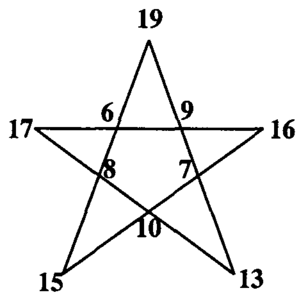
In this 4 by 4 Magic Square, some of the numbers are missing.

4) Copy the Magic Square and find the missing numbers.

5) What is the sum of each row, column, and diagonal?

16	**3**	**2**	
		11	**8**
9			**12**
4	**15**	**14**	**1**

A Magic Star works in a similar way:



At every intersection of lines there is a number.

6) What do you find about the total of the numbers on every diagonal?

7) Construct a Magic Star using the numbers 1, 2, 3, 4, 5 for the inside intersections and the numbers 5, 7, 8, 9, 11 for the outside ones (the total is 22).

11 Trigonometry (jyāgaṇitam)

Trigonometric Ratios (jyānayanam)

Now we shall see the source of these ratios.
Consider the (*prathama padam*), the first quadrant NE of the circle with centre 'O'. EQ is arc *jya*. Drop a perpendicular from Q to cut the east-west diameter NE at P. Now PQ is known as *bhujā*. Similarly QR is drawn perpendicular to NS and QR is called *koṭi* of the arc EQ.

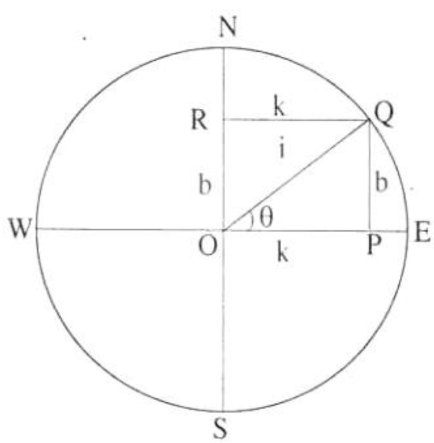
The radius *trijyā* is normally taken as 1 and let the *koti* RQ = k and *bhujā* PQ = b. Since OPQR is a rectangle, OP = k and OR = b.

Now

bhujā jyā $= \frac{PQ}{OQ}$ or $\frac{OR}{OQ}$

koṭi jyā $= \frac{RQ}{OQ}$ or $\frac{OP}{OQ}$

In the right angled triangle OPQ, if the measurement of angle POQ, which is also the degree measure of the arc EQ, is represented by the Greek letter Θ (Theta),, we have



Sin Θ $= \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{PQ}{OQ}$ = *bhujā jyā*

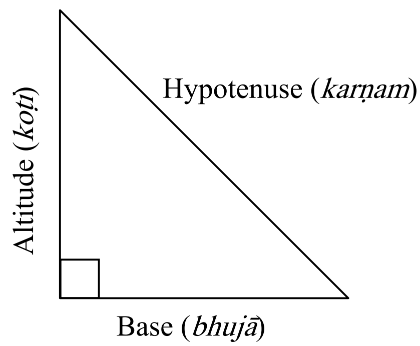
Cos Θ $= \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{OP}{OQ}$ = *koṭi jyā*

Tan Θ $= \frac{\text{opposite side}}{\text{adjacent side}} = \frac{PQ}{OP}$ = *tānaka jyā*

Cot Θ $= \frac{\text{adjacent side}}{\text{opposite side}} = \frac{OP}{PQ}$ = *kotitānaka jyā*

Later we will see how we can use this information. The main practical application will be to find positions of planets in astronomy. For now, just memorize the names and try to comprehend the relationship between an angle and a line.

Pythagoras Theorem (*bhujā-koṭi-karṇa-varga-nyāyam*)



There is natural phenomena hidden in a right angled triangle that has been used since very ancient times. The one which is used most is:

$(\text{hypotenuse})^2 = (\text{base})^2 + (\text{altitude})^2$

karṇavargam = *bhujāvargam* + *kotivargam*

From this formula, we can calculate one side by knowing the other sides. The beauty of this phenomena is that many objects can be divided into right angled triangles and then the *bhujā-koṭi-karṇa-varga-nyāyam* can be applied.

Later on we will see practical use of this phenomena while calculating positions of planets.

12 Calculations with numbers of base 60

As we learn in *Jyotish*, the stars in *rasi chakra* are divided into 12 *rasis* (signs). Since the *rasi* chakra covers the whole circle ($360°$), one *rasi* covers $360° / 12 = 30°$.

1° *kalā* (degree) = 60’ *ili*
1’ *ili* (minutes) = 60” *vili*
1” *vili* (second) = 60’” *talpara*

From the table above we can understand that while calculating with degrees, we are dealing with numbers with a base of 60. For us this is very important as we will always need this information to properly read Ephemeris, from where we will get positions of all the planets at any given time and place; from there we can actually make a chart. Right now it will be important for us be able to convert *kalā* to *ili*, *vili* and *talpara*, and to be able to add and subtract the figures.

Example 1

1) How many *ili* will be 10 *kalā*?
2) How many *vili* will be 10 *kalā*?

$10° \times 60 = 600'$ *ili*
$10° \times 60 \times 60 = 36000''$ *vili*

The difference between base 100 and base 60 is that base 100 has numbers from 0 – 99 and then we carry one to the place of hundreds. In base 60 we have numbers 0 – 59 and then instead of 60 we carry 1 to the next place.

Degrees and time is measured in numbers of base 60. So 1 hour has 60 minutes, 1 minute has 60 seconds. Similarly 1º has 60 minutes and 1 minute has 60 seconds. One complete circle has 6 × 60º degrees which is 360º.

<u>Addition</u>

Example 2

$20°20' + 10°10' = 30°30'$

When we end up with more then 60’ we have to subtract the same from the number and carry 1 to the next place.

$30°40' + 10°30' = 40°70'$
But as we said there is only 60 *ili* in a *kalā* so we subtract 60 from 70 and the result will be $41°10'$.

Example 3

Here we get a more complicated number to see how it functions:

(i)
$$\begin{array}{cccccc} 2 & 2^\circ & 3 & 9' & 5 & 2'' \\ 1 & 4^\circ & 2 & 7' & 4 & 5'' \\ \hline & & & & & 7'' \end{array}$$

We go from right to left, starting with seconds. Here we add 5 and 2. The place of units counts from 0-9. Then the result will be 7.

(ii)
$$\begin{array}{cccccc} 2 & 2^\circ & 3 & 9' & \dot{5} & 2'' \\ 1 & 4^\circ & 2 & 7' & 4 & 5'' \\ \hline & & & & 3 & 7'' \end{array}$$

In the next step, we have to be careful as the place holds digits 0-5. That means that if the result is greater then 5, we subtract 6, write the result under the number and put a dot above 5. Here 5 + 4 = 9. Then 9 – 6 = 3.

(iii)
$$\begin{array}{cccccc} 2 & 2^\circ & 3 & \dot{9}' & \dot{5} & 2'' \\ 1 & 4^\circ & 2 & 7' & 4 & 5'' \\ \hline & & & 7' & 3 & 7'' \end{array}$$

The minutes are like seconds. Then 9 + 7 = 16 + 1 = 17. Write 7 and put a dot.

(iv)
$$\begin{array}{cccccc} 2 & 2^\circ & \dot{3} & \dot{9}' & \dot{5} & 2'' \\ 1 & 4^\circ & 2 & 7' & 4 & 5'' \\ \hline & & 0 & 7' & 3 & 7'' \end{array}$$

Here 3 + 2 = 5 + 1 = 6. Again 6 being greater then 5 we subtract 6 from it. 6 – 6 = 0. Write 0 and carry 1.

(v)
$$\begin{array}{cccccc} 2 & 2^\circ & \dot{3} & \dot{9}' & \dot{5} & 2'' \\ 1 & 4^\circ & 2 & 7' & 4 & 5'' \\ \hline & 7^\circ & 0 & 7' & 3 & 7'' \end{array}$$

Now degrees. 2 + 4 = 6 + 1 = 7.

(vi)
$$\begin{array}{cccccc} 2 & 2^\circ & \dot{3} & \dot{9}' & \dot{5} & 2'' \\ 1 & 4^\circ & 2 & 7' & 4 & 5'' \\ \hline 3 & 7^\circ & 0 & 7' & 3 & 7'' \end{array}$$

2 + 1 = 3.

Subtraction:

Example 4

(i) 4	2°	3	6 '	5	$\dot{1}$ "
2	4°	5	3 '	3	5 "
					6 "

We go from right to left, starting with seconds. Here 5 is greater then 1 so we add 10 and subtract 5 from 11. 11 – 5 = 6 and put a dot above 1.

(ii) 4	2°	3	6 '	5	$\dot{1}$ "
2	4°	5	3 '	3	5 "
				1	6 "

In the next step, we have to remember that this place holds digits 0-5. 3 + 1 = 4 from previous calculation. 5 – 4 = 1.

(iii) 4	2°	3	6 '	5	$\dot{1}$ "
2	4°	5	3 '	3	5 "
			3 '	1	6 "

The minutes are like seconds. 6 – 3 = 3.

(iv) 4	2°	$\dot{3}$	6 '	5	$\dot{1}$ "
2	4°	5	3 '	3	5 "
		4	3 '	1	6 "

Here, 5 is greater then 3. Then we have to add 6 to 3 and get 9. Then we get 9 – 5 = 4. That is result and put dot.

(v) 4	$\dot{2}$°	$\dot{3}$	6 '	5	$\dot{1}$ "
2	4°	5	3 '	3	5 "
	7°	4	3 '	1	6 "

Now degrees. 4 + 1 = 5. 12 – 5 = 7. Put dot.

(vi) 4	$\dot{2}$°	$\dot{3}$	6 '	5	$\dot{1}$ "
2	4°	5	3 '	3	5 "
1	7°	4	3 '	1	6 "

4 – 3 = 1.

Exercise 1

Calculate the following:

a) 25° 30’ + 10° 20’ b) 30° 25’ + 20° 10’ c) 42° 15’ + 15° 50’ d) 50° 20’ – 30° 10’

e) 40° 40’ – 20° 50’ f) 45° 20’ + 20° 55’ g) 50° 20’ + 20° 18’ h) 65° 10’ + 95° 45’

Note that *kalā* (degrees) have value up to 360. 360° =0° So if we get in our result more then 360°, then we simply subtract 360 from that number till we get number smaller then 360.

Similarly when we get a negative number (e.g. -45°) we simply add 360° + (-45°) = 360 – 45 = 315°.

Exercise 2

Get exercises from your teacher.

13 By the completion or non-completion (*pūrṇāpūrṇābhyām*)

Suppose you had to add the numbers 19, 8, 1.

● Is there an easy way to add them up?

It is easier to add 19 and 1 first, because it completes a 20.
So the answer would be 28.

● Try 3 + 6 + 7 + 8 + 4. What is the easy way here?

Well you could pair up 3 and 7 which complete a 10, and also 6 and 4 which complete another 10.
This gives the answer as 28 again.

> This technique of looking for the wholeness is described by the *sūtra* "*By the Completion or Non-Completion*".

In the following exercise you may like to indicate the numbers being combined by joining them up.
For example $\overset{\frown}{19 + 3 + 11} + 2 = \underline{35}$

Exercise 1

Find an easy way of doing these sums:

a) 29 + 7 + 1 + 5 b) 16 + 3 + 6 + 17 c) 8 + 51 + 12 + 3

d) 37 + 7 + 21 + 13 e) 13 + 16 + 17 + 24 f) 73 + 46 + 54 + 8

g) 33 + 25 + 22 + 35 h) 18 + 13 + 14 + 23 i) 16 + 64 + 12 + 16

After some practice you may find you can quickly add a long string of numbers by this method.

Exercise 2

Complete the following addition sums:

a) 3 + 9 + 5 + 7 + 1 b) 27 + 15 + 23

c) 43 + 8 + 19 + 11 d) 32 + 15 + 8 + 4

e) 24 + 7 + 8 + 6 + 13 f) 6 + 33 + 24 + 17

g) 73 + 68 + 27 h) 41 + 16 + 23 + 9 + 21 + 18 + 9

i) 8 + 6 + 9 + 8 + 4 + 12 + 11 + 2 j) 17 + 12 + 13 + 9 + 21 + 18 + 9

k) 4 4	l) 3 5	m) 4 8	n) 6 3 2 7	o) 5 4 9
2 2	7 6	3 8	5 8 4	1 8 2
6 9	4 5 +	6 2	7 4 3 +	3 1 7
8 6 +		7 1 +		2 4 1
				7 2 6
				3 2 1 +

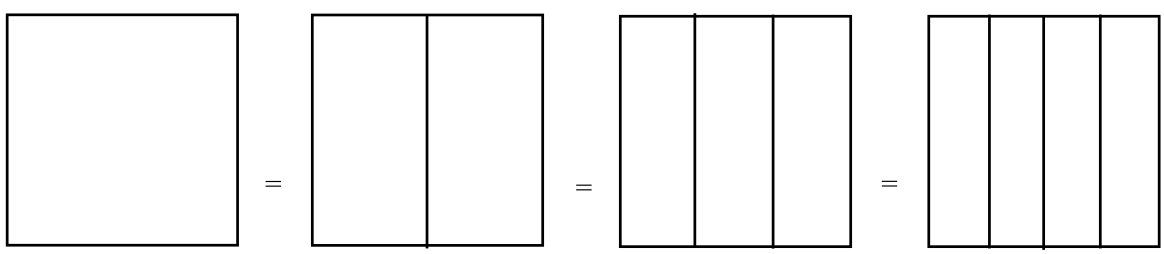
Completing The Whole- Fractions

When a whole one is divided we get fractions.
We can think of a whole one being divided into 2 halves or 3 thirds or 4 quarters, and so on.

We can write this as:

$1 = \frac{2}{2}$ $1 = \frac{3}{3}$ $1 = \frac{4}{4}$ etc.

or geometrically:



1) Copy out lines (1) and (2) above and continue the sequence up to the eighths.

Now suppose you have $\frac{3}{4}$ and you want to make it up to a whole one.

You will need $\frac{1}{4}$.

You can write $\frac{1}{4} + \frac{3}{4} = 1$

Exercise 3

Write out these sums but replace the question mark with the correct fraction:

a) $? + \frac{1}{2} = 1$ b) $? + \frac{2}{5} = 1$ c) $? + \frac{1}{3} = 1$ d) $? + \frac{5}{6} = 1$ e) $? + \frac{3}{8} = 1$

f) $? + \frac{7}{10} = 1$ g) $? + \frac{1}{7} = 1$ h) $? + \frac{3}{11} = 1$ i) $? + \frac{13}{20} = 1$ j) $? + \frac{99}{100} = 1$

Exercise 4:

Ask your teacher for more exercises.

Completing the whole-shapes

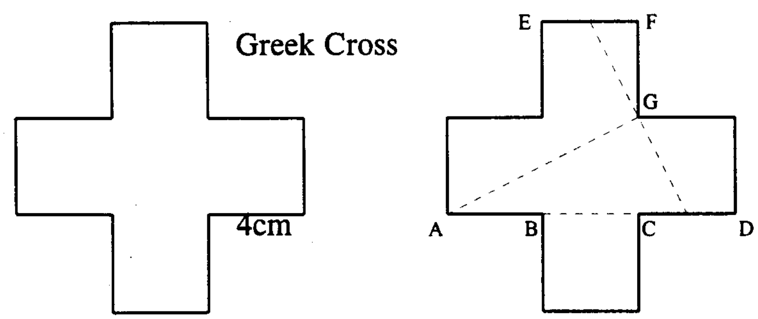
2) Here you are asked to solve three puzzles:

● Puzzle 1
Get the five pieces of Puzzle 1 and arrange them to form a dodecagon (12-sided figure). Next rearrange them to form a square.

● Puzzle 2
Get the six pieces of Puzzle 2 and make an octagon (8-sided figure) out of them. Then rearrange these pieces to form a square.

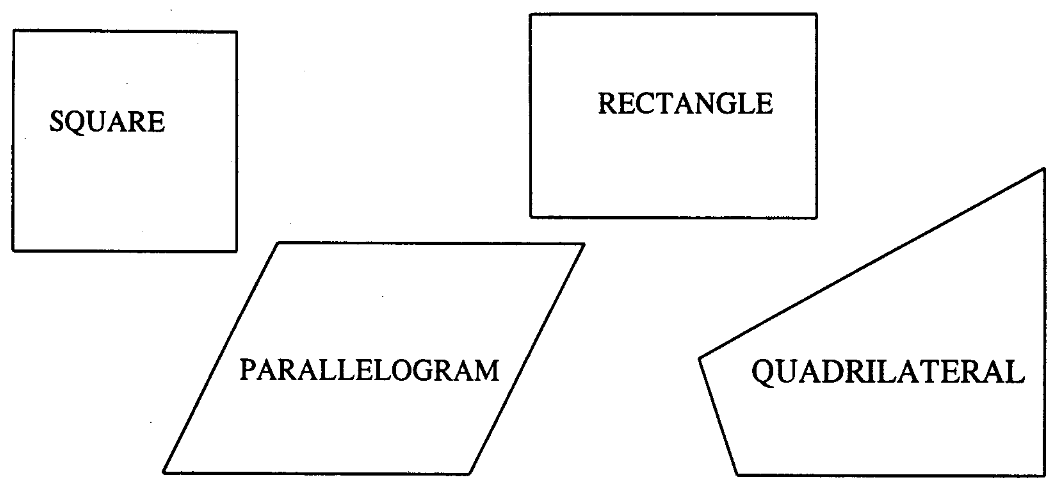
● Puzzle 3
For this puzzle you will need to make the 5 pieces yourself as shown below.

On squared or graph paper draw a Greek Cross with all sides 4cm



Draw the lines shown dashed above: B to C, A to G and the mid point of EF to the mid point of CD.

Then see if you can make each of the shapes shown below:



Now reconstruct the original Greek Cross.

14 General Multiplication

Now we will learn multiplication, by which any two numbers can be multiplied together in one line, by mere mental arithmetic. For that reason it is called general multiplication method. The *sūtra* which we are going to apply here is "Vertically and Crosswise" *ūrdhva tīryagbhyām*.

We know how to multiply single digits. Now let us see what happens if we multiply double digit with a single digit.

Example 1

Find 74 × 8.

We multiply each of the figures in 74 by 8 starting on the right:

4 × 8 = 32 and 7 × 8 = 56.

These are combined by carrying the 3 in 32 over to the 6 in 56: $5\overset{\frown}{6,3}2 = 592$

The inner figures are merged together. So 74 × 8 = 592.

Example 2

Find 56 × 6

The products are: 6 × 6 = 36 and 6 × 5 = 30

Then we combine them: $3\overset{\frown}{0,3}6 = 336$

Example 3

Find 827 × 3.

The three products are 24, 6, 21

The first two products are combined: 24, 6 = 246 and no carry here as 6 is a single figure, then 246 is combined with the 21: $24\overset{\frown}{6,2}1 = 2481$. So 827 × 3 = 2481.

Example 4

Find 77 × 4

The products are 28, 28

From there $2\overset{\frown}{8,2}8 = 308$ (the 28 is increased by 2 to 30). The result 77 × 4 = 308.

Checking result by *bījāṅkam* method

Similarly like for addition we can do a *bījāṅkam* check for multiplication.

Example 5

The previous multiplication:

$$\begin{array}{r} 7\ 7 \\ 4\ \times \\ \hline 3\ 0\ 8 \\ \hline \end{array} \qquad \text{Check:} \quad \begin{array}{l} 5 \\ 4\ \times \\ \hline 2 \\ \hline \end{array}$$

The *bījāṅkam* check has also been carried out above. The *bījāṅkam* of the numbers being multiplied are 5 and 4, and when these are multiplied we get 2. Since the *bījāṅkam* of the answer, 308, is also 2 this shows us that the answer is probably correct.

Example 6

$$\begin{array}{r} 6\ 2 \\ 4\ \times \\ \hline 2\ 4\ 8 \\ \hline \end{array} \qquad \text{Check:} \quad \begin{array}{l} 8 \\ 4\ \times \\ \hline 5 \\ \hline \end{array} \qquad (\text{since } 8 \times 4 = 32 \text{ and } 3 + 2 = 5)$$

Check that you agree that the check here confirms the answer, since the *bījāṅkam* of 248 is the same as the *bījāṅkam* of 8 × 4.

Exercise 1

Multiply the following mentally and do *bījāṅkam* check to see if the answer is correct:

a) 73 × 3 b) 63 × 7 c) 424 × 4 d) 777 × 3 e) 234 × 3

f) 654 × 3 g) 717 × 8 h) 876 × 7 i) 587 × 6j) 835 × 4

Now we will learn how to multiply by numbers of two and more digits. For that we will first explain the principle of the *ūrdhva tīryagbhyām sūtra*. We will start from the right most digit at the bottom row. That digit we will call a *nābhi*. Now we will multiply the *nābhi* with the target digit which we will call *śaravyam*. The *nābhi* always stays at the bottom row and the *śaravyam* will always be at the top row. They will move clockwise as a *chakra*.

Example 7

Find 21 × 23

Mentally set the numbers below each other like this:

```
2 1
2 3 x
-----
-----
```

There are 3 steps:

```
2   1
    ↑
2   3 x
-------
    3
-------
```

a) Here the 3 will be *nābhi* in the beginning. In the first step we multiply vertically (*ūrdhva*). So our *śaravyam* will be the digit above *nābhi* that is 1. So 3 x 1 = 3.

```
2   1
```

```
2   3 x
-------
  8 3
-------
```

b) Now we move the *śaravyam* one place to the left, that means the digit 2 on the top. 3 x 2 = 6. Now the charka moves clockwise an we get to multiply 2 x 1 = 2. We write down the sum of these two results which is 6 + 2 = 8.

```
2   1
↑
2   3 x
-------
4 8 3
-------
```

c) As there are no more numbers on the left of the *śaravyam*, we will move the *nābhi*. Now the 2 on the bottom will be the *nābhi* and the 2 on top will be *śaravyam*. 2 x 2 = 4. So 21 x 23 = 483.

Example 8

Find 11 × 32

```
1   1
3   2 x
-------
3 5 2
-------
```

a) Set 2 as *nābhi* and 1 on the right as *śaravyam.* 2 × 1 = 2.

b) 2 stays as *nābhi* and 1 on the left becomes *śaravyam*. 2 × 1 = 2. Then move the *chakra* clockwise and calculate 3 × 1 = 3. The sum is 2 + 3 = 5.

c) Now we set the 3 as *nābhi* and 1 as *śaravyam*. 3 × 1 = 3. Wrute down 3. So 11 × 32 = 352.

This is of course very easy and straightforward, and we should now practice this vertical and cross-wise pattern to establish the method.

Exercise 2

Multiply mentally:

a)	b)	c)	d)	e)	f)	g)
2 2	2 1	2 1	2 2	6 1	3 2	3 1
3 1 x	3 1 x	2 2 x	1 3 x	3 1 x	2 1 x	3 1 x
______	______	______	______	______	______	______

The previous Examples involved no carry figures, so let us consider this next.

Example 9

Find 23 × 41

$$\begin{array}{ccc} 2 & & 3 \\ 4 & & 1 \\ \hline 9 & {}_{1}4 & 3 \\ \hline \end{array}$$

The three steps give us: $1 \times 3 = 3$,
$1 \times 2 + 4 \times 3 = 14$,
$4 \times 2 = 8$

Here the 14 involves a carry figure, so in building up the answer mentally from the right we merge these numbers as before.

The mental steps are: 3
14,3 = 143
$\overset{\frown}{8,1}43 = 943$ (the 1 is carried over to the left)

So 23 × 41 = 943

Example 10

Find 23 × 34

$$\begin{array}{cccc} 2 & & 3 & \\ 3 & & 4 & x \\ \hline 7 & {}_{1}8 & {}_{1}2 & \\ \hline \end{array}$$

The steps are : 12
$1\overset{\frown}{7,1}2 = 182$
$\overset{\frown}{6,1}82 = 782$

Example 11

Find 33 × 44.

```
   3   3
   4   4 x
 1 4 5 2
```

The steps are: 12

$\overset{\frown}{24,12} = 252$

$\overset{\frown}{12,252} = 1452$

We can now multiply any two 2-figure numbers together in one line. This is clearly an application of the Vertically and Cross-wise Sutra.

Exercise 3

Multiply the following mentally:

```
a) 2 1    b) 2 3    c) 2 4    d) 2 2    e) 2 2    f) 3 1
   4 7 x     4 3 x     2 9 x     2 8 x     5 3 x     3 6 x

g) 2 2    h) 3 1    i) 4 4    j) 3 3    k) 3 3    l) 3 4
   5 6 x     7 2 x     5 3 x     8 4 x     6 9 x     4 2 x

m) 3 3    n) 2 2    o) 3 4    p) 5 1    q) 3 5    r) 5 5
   3 4 x     5 2 x     6 6 x     5 4 x     6 7 x     5 9 x

s) 5 4    t) 5 5    u) 4 4    v) 4 5    w) 4 8    x) 3 4
   6 4 x     6 3 x     8 1 x     8 1 x     7 2 x     1 9 x
```

y) Can you see how this method simplifies when:
i) both numbers end in 1
ii) the last figures of the numbers, or the first figures, or both figures of one number, are the same?

You may have found in this exercise that you prefer to start with the cross-wise multiplications, and put the left and right vertical multiplications on afterwards.

Explanation It is easy to understand how this method works. The vertical product on the right multiplies units by units and so gives the number of units in the answer. The cross-wise operation multiplies tens by units and units by tens and so gives the number of tens in the answer. And the vertical product on the left multiplies tens by tens and gives the number of hundreds in the answer.

Multiplying 3-Figure Numbers

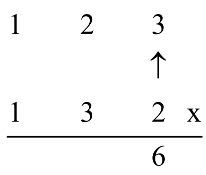
For multiplying 3 numbers we extend the vertical/cross-wise/vertical pattern by additional 2 steps. The *nābhi* keeps moving from the right to the left and the *śaravyam* moves from the left to the right.

Example 12

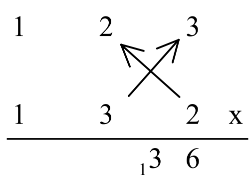
Find 123 × 132.

```
    1 2 3
    1 3 2  x
---------
1 6 2 3 6
---------
```

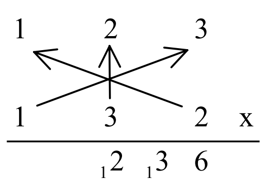
For multiplying of three digits numbers we require 5 steps:



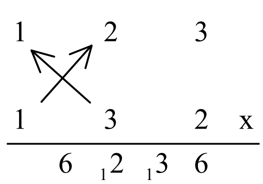
a) *Nābhi* is 2 and *śaravyam* is 3. 2 x 3 = 6.



b) *Nābhi* is 2 and *śaravyam* is 2. 2 x 2 = 4. Then move the *chakra* and calculate 3 x 3 = 9. The sum is 4 + 9 = 13.



c) *Nābhi* is 2 and *śaravyam* is 1. Then we move the *chakra* two times clock-wise. 2 x 1 + 3 x 2 + 1 x 3 = 11 + 1 =



d) There are no more steps to be calculated for *nābhi* 2, so now 3 becomes the *nābhi* and 1 *śaravyam*. 3 x 1 + 1 x 2 = 5 + 1 = 6.

```
1      2      3
↑
1      3      2   x
--------------------
1   6  ₁2  ₁3  6
--------------------
```

e) In the last step 1 becomes *nābhi* and the other 1 is *śaravyam*. 1 x 1 = 2 and there are no carries from previous step. So the result 123 x 132 = 16236.

But there is more elegant solution:

We can split the numbers into 12/3 and 13/2, treating the 12 and 13 as if they were single figures:

```
12        3
13        2    x
-----------------
162  ₆3   6
-----------------
```

a) *nābhi* 2, *śaravyam* 3	2 × 3 = 6	6
b) *nābhi* 2, *śaravyam* 12 and move the *chakra*	2 × 12 + 13 × 3 = 63	63,6 = 636
c) *nābhi* 13, *śaravyam* 12	13 × 12 = 156 + 6 = 162	156, 636 = 16236

Exercise 4

Multiply, treating the numbers as 2-figure numbers:

a)	b)	c)	d)	e)	f)
1 1 2	1 2 3	1 2 3	1 1 2	4 2 1	2 2 1
2 0 3	1 3 1	1 2 2	1 2 3	2 2	1 2 2
______	______	______	______	______	______

Example 13

304 × 412 = 125248

Here we may decide to split the numbers after the first figure: 3/04 × 4/12.

```
 3     04
 4     12
----------
12  52  48
----------
```

The 3 steps are:

a) 12 × 4 = 48
b) 12 × 3 + 4 × 4 = 52
c) 4 × 3 = 12

These give the 3 pairs of the figures in the answer.

> When we split the numbers in this way the answer appears **two digits at a time**.

Exercise 5

Multiply using pairs of digits:

a)	b)	c)	d)	e)	f)
2 1 1	3 0 7	2 0 3	2 1 1	5 0 4	5 0 1
3 0 4	4 0 7	4 3 2	3 1 1	5 0 4	5 0 1
______	______	______	______	______	______

Multiplying from Left to Right

It is a good habit to check if our answer are correct. One of the methods is to do the calculation again, but this time from left to right. The left to right calculation is also used in case when we want to know only the most significant digits of the result.

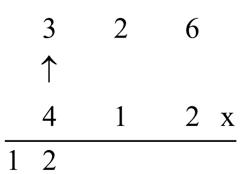
The method is very similar to the calculation from right to left with the only difference that we set the *nābhi* as the digit on the left, bottom and the *śaravyam* will be the digit above *nābhi*. The *chakra* will be moving counter clock-wise.

Example 14

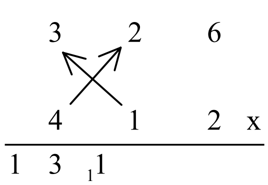
Find 326 × 412.

```
      3 2 6
      4 1 2 x
-----------
1 3 4 3 1 2
-----------
```

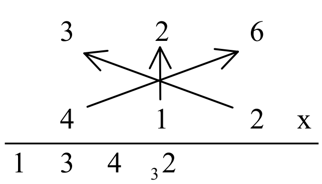
For multiplying of three digits numbers we require 5 steps:



a) *Nābhi* is 4 and *śaravyam* is 3. 4 x 3 = 12.



b) *Nābhi* is 4 and *śaravyam* is 2. 4 x 2 = 8. Then move the *chakra* and calculate 1 x 3 = 3. The sum is 8 + 3 = 11 and 12, 11 = 131



c) *Nābhi* is 4 and *śaravyam* is 6. Then we move the *chakra* two times clock-wise. 4 x 6 + 1 x 2 + 2 x 3 = 32. Then 131, 32 = 1342

```
3    2    6
      ↖↗
      ↗↖
4    1    2  x
--------------
1 3 4 3 ₁0
--------------
```

d) There are no more steps to be calculated for *nābhi* 4, so now 1 becomes the *nābhi* and 6 *śaravyam*. 1 x 6 + 2 x 2 = 10. So 1342, 10 = 13430

```
3    2    6
          ↑
4    1    2  x
--------------
1 3 4 3 1 ₁2
--------------
```

e) In the last step 2 becomes *nābhi* and 6 is *śaravyam*. 2 x 6 = 12. Then 13430, 12 = <u>134312</u>, and that is the result of 326 x 412

Exercise 6

Multiply the following mentally first left to right then right to left to verify the answer:

```
a) 3 3     b) 4 7     c) 1 5     d) 3 2     e) 2 2     f) 4 1
   5 2 x      3 1 x      4 2 x      4 1 x      2 5 x      2 7 x
   _____      _____      _____      _____      _____      _____

   _____      _____      _____      _____      _____      _____

g) 3 1     h) 3 1     i) 3 8     j) 3 4     k) 3 5     l) 5 3
   4 2 x      6 3 x      4 3 x      6 9 x      4 8 x      6 8 x
   _____      _____      _____      _____      _____      _____

   _____      _____      _____      _____      _____      _____
```

<u>Multiplying Of Numbers By Not Completion:</u>

In multiplying a long number by a number with lesser digits we put the longer one on the top line and the shorter one under the longer one. Due to the fact that the shorter number has less digits, the *chakra* does not always complete the full circle.

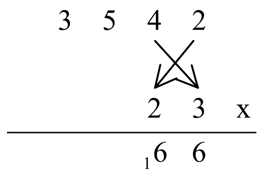
Example 15

Find 3542 × 23

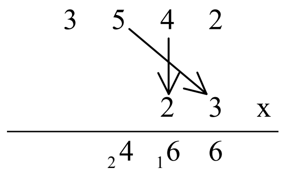
```
 3 5 4 2
     2 3 x
----------
8 1 4 6 6
----------
```

```
 3 5 4 2
       ↓
     2 3 x
----------
       6
```

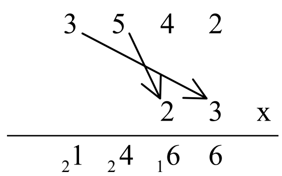
a) The 2 is *nābhi* and 3 is *śaravyam*. 2 x 3 = 6.



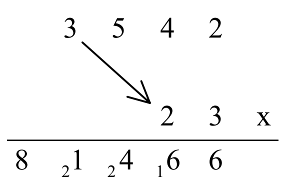
b) 2 is *nābhi* and the other 2 is *śaravyam*. Then the *chakra* moves counter clock-wise. 2 x 2 + 4 x 3 = 16.



c) *nābhi* is 2, but *śaravyam* would be 0 so we skip this calculation and move the *chakra*. 4 x 2 + 5 x 3 = 23 + 1 = 24.



d) *nābhi* is still 2 but the *śaravyam* is again 0, so we move the *chakra* till we get the result. 5 x 2 + 3 x 3 = 19 + 2 = 21.



e) Here the 4 becomes the *nābhi*, but as there are no digits for the *śaravyam* we keep moving the *chakra* and get 3 x 2 = 6 + 2 = 8. At this point there are no more calculations to be done, since there are no more digits for the *śaravyam*. So 3542 x 23 = 81466.

Moving Multiplier

Another variation of multiplying a long number by a single figure is a method called moving multiplier. For Example 4321 × 2, we multiply each of the figures in the long number by the single figure. We may think of the 2 moving along the row, multiplying each figure vertically by 2 as it goes.

Example 16

Find 4321 × 32

```
4 3 2 1
3 2
```

Similarly here we put 32 first of all at the extreme left.
Then vertically on the left, 4x3=12
And cross-wise, 4x2+3x3=17

```
4 3 2 1
  3 2
```

Then move the 32 along and multiply cross-wise:
3x2+2x3=12

```
4 3 2 1
    3 2
```

Moving the 32 once again:
Multiply cross-wise: 2 × 2 + 1 × 3 = 7
Finally the vertical product on the right is 1 × 2 = 2

These five results (in bold), 12,17,12,7,2 are combined mentally, as they are obtained, in the usual way:

$1\overparen{2, 1}7 = 137$

$13\overparen{7, 1}2 = 1382$

1382,7,2 = <u>138272</u>

So we multiply cross-wise in every position, but we multiply vertically also at the very beginning and at the very end.

Example 17

31013 × 21

Here the 21 takes the positions:

```
3 1 0 1 3      3 1 0 1 3      3 1 0 1 3      3 1 0 1 3
2 1              2 1              2 1              2 1
```

The 6 mental steps give: 6,5,1,2,7,3
So the answer is <u>651273</u>.

We get the same result if we start from the right, just the steps will be reversed and we have to write down the digits of the result starting from the right.

Exercise 7

Multiply using the not completion method from left to right and then from right to left:

a)	b)	c)	d)	e)	f)
3 2 1	3 2 1	4 2 1	3 2 1	1 2 1 2	1 3 3 1
2 1	2 3	2 2	4 1	2 1	2 2
____	____	____	____	____	____

Exercise 8

Multiply using the moving multiplier method from left to right and then from right to left:

a)	b)	c)	d)	e)	f)
2 3 1	4 2 1	1 2 4	1 4 1	1 2 3 2	1 4 3 1
3 2	2 4	3 2	3 3	2 2	2 1
____	____	____	____	____	____

Exercise 9

Ask your teacher for more exercises.

15 Straight Division

Now we will learn a division of numbers. It is nothing else then multiplication. We are trying to find out how many times a number fits into another number. There are many methods of division and each of them is applied in different circumstances.

However the straight division method is general division method by which any numbers of any size can be divided in one line. This come under the "Vertically and Cross-wise" and "On the Flag" *sūtras*.

The way how do we write the numbers down:

$456 \div 34$ is written as $3^{4}) \, 356$
$45752 \div 473$ is written as $4^{73}) \, 45752$
$782745 \div 6294$ is written as $6^{294}) \, 782745$

Dividend is a number which is being divided. **Divisor** is a number by which dividend is being divided.

Example 1

Suppose we want to divide 815 by 5

We set the sum out like this:

$$
\begin{array}{l|lll}
5) & 8 & {}_{3}1 & {}_{1}5 \\
\hline
 & 1 & 6 & 3
\end{array}
$$

We need to know how many 5s there are in 815.
Looking at the first figures we see that 5 goes into 8 1time. So set the 1 as the first digit of the answer.
Now we multiply the first digit of the answer by the divisor and subtract it from the first number of the dividend. $8 - 1 \times 5 = 3$.
Carry this 3 to the left of the next dividend digit and get the next dividend 31. Try how many 5's are in 31. We find 6 and that is second digit of answer. Now again $31 - 6 \times 5 = 1$. So carry 1 to the left of the next digit 5 and try 5's in 15. Find 3 and that is 3rd digit of the answer. $15 - 3 \times 5 = 0$ so there is no reminder (written R). The answer $815 \div 5 = 163$, R = 0

Example 2

Find $1638 \div 6$

$$
\begin{array}{l|llll}
6) & 1 & 6 & {}_{4}3 & {}_{1}8 \\
\hline
 & & 2 & 7 & 3
\end{array}
$$

Here the first digit of the dividend is smaller then the divisor. So we take the next digit also and try 6 in 16. We get 2 as the first answer digit. $16 - 2 \times 6 = 4$. So next dividend is 43. We get 7 as the second answer digit. $43 - 6 \times 7 = 1$. Now try 6 in 18. Since 18 is exactly divisible by 6 we get the 3rd digit of the answer 3 and reminder is 0.

For our convenience we can use D1, D2, D3 for dividends, A1, A2, A3 for answer digits and R1, R2, R3 for reminders.

Example 3

Find 22967 ÷ 7

$$\begin{array}{l} 7)\ \underline{2\ \ 2\ \ {}_{1}9\ \ {}_{5}6\ \ {}_{0}7} \\ \qquad\ \ 3\ \ 2\ \ 8\ \ 1 \end{array}$$

So <u>22967 ÷ 7 = 3281, R = 0</u>

a) D1 = 22
   A1 = 3, R1 = 1
b) D2 = 19
   A2 = 2, R2 = 5
c) D3 = 56
   A3 = 8, R3 = 0
d) D4 = 7
   A4 = 1, R4 = 0

Example 4

Find 36816 ÷ 8

$$\begin{array}{l} 8)\ \underline{3\ \ 6\ \ {}_{4}8\ \ {}_{0}1\ \ {}_{1}6} \\ \qquad\ \ 4\ \ 6\ \ 0\ \ 2 \end{array}$$

So <u>36816 ÷ 8 = 4602, R = 0</u>

a) D1 = 36
   A1 = 4, R1 = 4
b) D2 = 48
   A2 = 6, R2 = 0
c) D3 = 1
   A3 = 0, R3 = 1
d) D4 = 16
   A4 = 2, R4 = 0

Example 5

Find 41197 ÷ 8

$$\begin{array}{l} 8)\ \underline{4\ \ 1\ \ {}_{1}1\ \ {}_{3}9\ \ {}_{7}7\ \ {}_{5}} \\ \qquad\ \ 5\ \ 1\ \ 4\ \ 9 \end{array}$$

So <u>41197 ÷ 8 = 5149, R = 5</u>

a) D1 = 41
   A1 = 5, R1 = 1
b) D2 = 11
   A2 = 1, R2 = 3
c) D3 = 39
   A3 = 4, R3 = 7
d) D4 = 77
   A4 = 9, R4 = 5

Exercise 1

a) 632 ÷ 4 b) 582 ÷ 7 c) 3452 ÷ 5 d) 842 ÷ 6 e) 9352 ÷ 8 f) 3729 ÷ 3
g) 6243 ÷ 9 h) 478 ÷ 6 i) 2963 ÷ 2 j) 9534 ÷ 4 k) 73923 ÷ 5 l) 482345 ÷ 7

<u>The Bījāṅkam Check For Division</u>

The similar check by *bījāṅkam* as we did for multiplication can be applied here.

Take the example 5 above.

41197 ÷ 8 = 5149, R=5. From here we can assume that 5149 × 8 + 5 = 41197. Here we can apply *bījāṅkam* check. 5149 × 8 + 5 = 41197 → 1 × 8 + 5 = 4 → 4 = 4.

So *bījāṅkam* check agrees and the result is probably correct.

Exercise 2

Do *bījāṅkam* check for the calculations in exercise 1.

Calculation of decimal places:

Example 6

Find 3411 ÷ 8

$$8)\ \underline{3\quad 4\quad {}_{2}1\quad {}_{5}1\ /\ {}_{3}0\quad {}_{6}0\quad {}_{4}0}$$
$$4\quad 2\quad 6\ .\ 3\quad 7\quad 5$$

After last digit of the dividend insert zeros and continue division to get decimal places.

So <u>3411 ÷ 8 = 426.375</u>

Example 7

Find 2553 ÷ 6 to its decimal places

$$6)\ \underline{2\quad 5\quad {}_{1}5\quad {}_{3}5\ /\ {}_{3}0}$$
$$4\quad 2\quad 5\ .\ 5$$

So <u>2553 ÷ 6 = 425.5</u>

Exercise 3

Find the divisions to their decimal places:

a) 785 ÷ 4 b) 894 ÷ 5 c) 963 ÷ 6 d) 2458 ÷ 7 e) 5381 ÷ 8 f) 6987 ÷ 9

Division By Two Digit Numbers

Find 1472 ÷ 46

$$4^{6})\ \underline{1\quad 4\quad {}_{2}7\ /\ {}_{1}2}$$
$$3\quad 2\ /\ 0$$

So <u>1472 ÷ 46 = 32, R = 0</u>

a) 14 ÷ 4 gives A1 = 3 and R1 = 2
b) 27 – (3 x 6) = 27 – 18 = 9 so D2 = 9
   9 ÷ 4 gives A2 = 2 and R2 = 1
c) 12 – (2 x 6) = 12 – 12 = 0 and D3 = 0
   0 ÷ 4 gives A3 = 0 and R3 = 0

Here the actual divisor is 46, but we are going to divide by the first digit 4 only. Keep the next digit 6 as flag digit. Since there is one flag digit set apart the last digit of the dividend as remainder part.

This is straight division in one-line. In each step we have to subtract the sequential cross product between the quotient and flag digit, before actual division by the first digit.

Example 8

Find 5440 ÷ 85

$$8^5)\ \underline{5\quad 4\quad {}_{6}4\quad /\quad {}_{2}0}$$
$$\quad\ 6\quad 4\quad /\quad 0$$

So <u>5440 ÷ 85 = 64, R = 0</u>

a) 54 ÷ 8 → A1 = 6, R1 = 6
b) 64 – (6 x 5) = 64 – 30 =34
34 ÷ 8 → A2 = 4, R2 = 2
c) 20 – (4 x 5) = 20 – 20 = 0
0 ÷ 8 → A3 = 0, R3 = 0

Example 9

Find 3922 ÷ 74

$$7^4)\ \underline{3\quad 9\quad {}_{4}2\quad /\quad {}_{1}2}$$
$$\quad\ 5\quad 3\quad /\quad 0$$

So <u>3922 ÷ 74 = 53, R = 0</u>

a) 39 ÷ 7 → A1 = 5, R1 = 4
b) 42 – (5 x 4) = 22
22 ÷ 7 → A2 = 3, R = 1
c) 12 – (3 x 4) = 12 – 12 = 0
0 ÷ 7 = 0

Exercise 4

a) 457 ÷ 63 b) 7512 ÷ 37 c) 53148 ÷ 43 d) 64861 ÷ 52 e) 58295 ÷ 47 f) 116365 ÷ 34

> In case the dividend is lesser than the cross product we have to reduce the previous quotient suitably.

Example 10

Find 35724 ÷ 52

$$5^2)\ \underline{3\quad 5\quad {}_{5}7\quad {}_{5}2\quad /\quad {}_{1}4}$$
$$\quad\ 6\quad 8\quad 7\quad /\quad 0$$

So <u>35724 ÷ 52 = 687, R = 0</u>.

a) Get 35 ÷ 5. We see that it is 7 times and the reminder 0. So the next dividend is only 7. But we have to subtract the cross product 7 x 2 = 14 from 7. Therefore lessen the number of times to 6 instead of 7.
35 ÷ 5 → A1 = 6, R1 = 5
b) 57 – (6 x 2) = 57 – 12 = 45
Here also 5 goes 9 times, but we reduce it by one and take the quotient as 8 and carry 5
A2 = 8, R = 5
c) 52 – (8 x 2) = 52 – 16 = 36
36 ÷ 5 → A3 = 7, R3 = 1
d) 14 – (7 x 2) = 14 – 14 = 0
0 ÷ 5 = 0

Example 11

Find 37950 ÷ 46

$$4^6)\ \underline{3\quad 7\quad {}_{5}9\quad {}_{3}5\quad /\quad {}_{3}0}$$
$$\quad\ 8\quad 2\quad 5\quad /\quad 0$$

So <u>37950 ÷ 46 = 825, R = 0</u>

a) 37 ÷ 4 → A1 = 8, R1 = 5
b) 59 – (6 x 8) = 59 – 48 = 11
11 ÷ 4 → A2 = 2, R2 = 3
c) 35 – (2 x 6) = 35 – 12 = 23
23 ÷ 4 → A3 = 5, R3 = 3
d) 30 – (6 x 5) = 30 – 30 = 0

Exercise 5

a) $6552 \div 36$ b) $130295 \div 55$ c) $43476 \div 87$ d) $257838 \div 49$ e) $859196 \div 62$ f) $24638 \div 97$

Exercise 6

Ask your teacher for more exercises.

16 Doubling And Halving

Doubling and halving numbers is very easy to do and can help us in many ways. In the following exercise just write down the answers to the sums.

Exercise 1

Double the following numbers:

a) 43 b) 1234 c) 17 d) 71 e) 77 f) 707 g) 95 h) 59 i) 38

Halve the following numbers:

j) 64 k) 820 l) 36 m) 52 n) 94 o) 126 p) 234 q) 416 r) 57

> Since $4 = 2 \times 2$ we can multiply a number by 4 by doubling the number and doubling again the answer.

Example 1

Find 35×4

We simply double 35 to 70, then double 70 to 140.
So $\underline{35 \times 4 = 140}$.

> Similarly since $8 = 2 \times 2 \times 2$ we can multiply by 8 by doubling three times.

Find 26×8

Doubling 26 gives 52, doubling 52 gives 104 and doubling 104 gives 208.
So $\underline{26 \times 8 = 208}$.

Exercise 2

Multiply the following:

a) 53×4 b) 28×4 c) 33×4 d) 61×4 e) 18×4 f) 81×4

g) 73×2 h) 16×8 i) 22×8 j) 45×8 k) 37×4 l) 76×8

The next exercise involves multiplying some fractions.

Exercise 3

Multiply the following:
a) 8 ½ × 4 b) 11 ½ × 8 c) 19 ½ × 4 d) 2 ¼ × 4 e) 5 ½ × 8 f) 9 ½ × 4

g) 7 ½ × 4 h) 16 ½ × 4 i) 3 ½ × 8j) 30 ½ × 4 k) 3 ¼ × 4 l) 9 ¼ × 8

The answers to these 12 sums can be converted to a special message according to the following scheme.

Convert each answer to its *bījāṅkam* and convert each *bījāṅkam* to a letter using the following code:

D	E	L	N	O	R	V	W	Y
1	2	3	4	5	6	7	8	9

Halving numbers is something which can also be repeated.

So if for example we halved a number and then halved it again we would be dividing the number by 4.

Example 3

Divide 72 by 4.

We halve 72 twice: half of 72 is 36 and half of 36 is 18.
So $\underline{72 \div 4 = 18}$.

Example 4

Divide 104 by 8

Here we halve three times:
Half of 104 is 52, half of 52 is 26 and half of 26 is 13.

So $\underline{104 \div 8 = 13}$

Exercise 4

Divide by 4:

a) 56 b) 68 c) 84 d) 180 e) 116 f) 92 g) 34

Divide by 8:

h) 120 i) 440 j) 248 k) 216 l) 44 m) 64 n) 888

Extending The Multiplication Tables

You may have noticed another way of doing some of the sums in Exercise 2.
For example, for 18 × 4 you may have thought that since you know that 9 × 4 = 36, then 18 × 4 must be double this, which is 72.

Similarly if you don't know 8 × 7 but you do know that 4 × 7 = 28, you can just double 28. So 8 × 7 = 56.

The following examples assume you know your tables up to 10 × 10, but if you don't know all these you should still be able to find your way to the answer.

Example 5

Find 6 × 14

Since we know that 6 × 7 = 42, it follows that 6 × 14 = 84.

Find 14 × 18.

Halving 14 and 18 gives 7 and 9, and since 7 × 9 = 63 we double this twice.
We get 126 and 252, so 14 × 18 = 252.

Exercise 5

Multiply the following:

a) 16 × 7 b) 18 × 6 c) 14 × 7 d) 12 × 9 e) 4 × 14 f) 6 × 16

g) 7 × 18 h) 9 × 14 i) 16 × 18j) 14 × 16k) 12 × 18 l) 16 × 12

All this comes under the Vedic *sūtra anurūpyena*, which means Proportionately. We will be seeing many other applications of this formula later on.

Exercise 6

Get more exercises from your teacher.

17 Divisibility

<u>Divisibility By 2, 5,10</u>

Look at the following sequence of numbers:

2, 4, 6, 8, 10, 12, 14, 16......

1) What factor apart from 1 is common to all these numbers?
   (Look at your table of factors from chapter 7 if necessary)

2) How can you tell if 2 is a factor of a number?

3) Does every even (*yugma*) number have 2 as a factor?

4) Do any odd (*ayugma*) numbers have 2 as a factor?

All numbers ending in an even (*yugma*) number or zero must have 2 as a factor. We say that the number is **divisible** by 2. We can also say that any number ending in an even (*yugma*) number or zero is a multiple of 2.

So, given the number 38, we may say: 2 is a factor of 38
or 38 is a multiple of 2
or 38 is divisible by 2.

Look at the following sequence:

5, 10, 15, 20, 25, 30......

5) Which factor apart from one is common to all these numbers?

All numbers ending in 5 or 0 are divisible by 5.

What about the following series of numbers?

10, 20, 30, 40, 50, 60....

6) What number apart from one, two or five divides into all of the above?

All numbers ending in 0 are divisible by 10.

The Vedic formula which applies for testing for divisibility by 2, 5 and 10 is the sub-*sūtra* "Only the Last Terms" – *antayor eva*.

Exercise 1

Which of the following numbers are divisible by 2 or 5 or 10?

a) 12 b) 17 c) 15 d) 20 e) 28 f) 70 g) 61 h) 124 i) 145 j) 16750

k) 45368 l) 63425 m) 3540 n) 2364 o) 7353 p) 4829315 q) 8493075630

Exercise 2

Find a 2-figure number, divisible by ---

a) 2, whose figures add up to 5. (3 answers)
b) 2, whose figures add up to 7. (4)
c) 2, when the first figure is 3 times the last.
d) 2, when the first figure is 6 less than the last.
e) 2, when the product of the figures is 12. (3)
f) 2, when the product of the figures is 2.
g) 2, when the difference between the figures is 7. (3)
h) 2, when the difference between the figures is 8.
i) 2, when the last figure is 6 times the first.
j) 2, when the difference between the figures is 3. (7)
k) 5, if the product of the figures is 35.
l) 5, if one figure is greater than the other by 7.

Divisibility By 3 And 9

Look again at your 9-point circle.

7) Write down all the numbers you put on your '9'-branch. Write down what you think the next three numbers should be.

8) Do you notice anything? Where have you seen this set of numbers before?

All numbers whose *bījāṅkam* is 9 are divisible by 9.

Look again at your 9-point circle.

9) Write down all the numbers on your 3, 6 and 9 branches in order with the smallest first.

10) Do you notice anything? Where have you seen this set of numbers before?

All numbers with digit sum of 3, 6 or 9 are divisible by 3.

Summary

We can summarize the divisibility tests for 2, 3, 5, 9, 10 as follows:

For 2: is the last figure even or 0?
For 3: is the *bījāṅkam* 3, 6 or 9?
For 5: is the last figure 5 or 0?
For 9: is the *bījāṅkam* 9?
For 10: is the last figure 0?

If the answer is yes the number is divisible, otherwise it is not divisible.

11) Copy and complete the following table:

Number	Divisible by 2	Divisible by 3	Divisible by 5	Divisible by 9	Divisible by 10
12	yes	yes	no	no	no
15	no		yes		
18					
20					
30					
36					
45					
50					
12825					
645840					
71023					
66273					

Exercise 3

Find a 2-figure number, divisible by ---

a) 3, if the first figure is 2. (3 answers)
b) 3, if the first figure is 7. (3)
c) 3, if the product of the figures is 14. (2)
d) 3, when the first figure is 4, and the difference between the figures is 2.
e) 3, when one figure is 5, and the difference is 4. (2)
f) 3, when one figure is 8, and the other is odd. (4)
g) 3, when both figures are even, and their difference is 0.
h) 3, when the first figure is odd and their sum is 6. (3)
i) 3, when the last figure is 3 times the first.
j) 3, when the last figure is half the first. (4)
k) 3, when the last figure is 4. (3)
l) 3, if both figures are even. (5)
m) 5, if the difference between the figures is 4. (3)
n) 5, if both figures are odd, and the first is greater than the last. (2)
o) 5, if both figures are odd and their difference is 4. (2)
p) 5, if only one figure is odd, and their difference is 3. (3)
q) 2, when the last figure is half the first. (2)
r) 2, when the product of the figures is 28.

s) 9, if the first figure is twice the second.
t) 9, if both figures are prime. (2)
u) 10, if the *bījāṅkam* is 7.
v) 6 and 10. (3)

Divisibility By 4

We saw earlier that we can tell if a number can be divided exactly by 2 by looking at the last figure- if it is even or 0 then it can be divided by 2.

This works because any number is a multiple of ten plus its last figure.

For example 34 = 30 + 4, 543 = 540 + 3.

And since 2 will always divide into the first part (30 and 540 above) the full number will be divisible by 2 if the last figure is. So 34 is divisible by 2 and 543 is not.

We can extend this idea:

If 4 divides into the last two figures of a number then 4 divides into the whole number.

Example 1

In the number **1234** we can ignore the 12 at the beginning and see if 4 divides into 34.
We can do this by halving 34 twice: half of 34 is 17, and half of 17 is not a whole number, so we can say that <u>1234 is not divisible by 4</u>.

Example 2

Is 3456 divisible by 4?

Just look at the last two figures, 56. Can 56 be halved twice without going into fractions?

Half of 56 is 28 and half of 28 is 14, so the answer is yes,
<u>3456 can be divided by 4.</u>

The reason this method works is that 3456 = 3400 + 56 and 4 must divide into 3400 (because 4 divides into 100) so we only need to find out if 4 divides into the last two figures.

Exercise 4

Test the following numbers for divisibility by 4 (just write yes or no):

a) 48084	b) 468	c) 112233	d) 7007	e) 6666	f) 99008
g) 7654	h) 74	i) 82182	j) 55000	k) 19181716	l) 949596

This method comes under the Vedic *sūtra – anurūpyena* which means Proportionately, but another *sūtra* can be used to do the same job.

This is *sopāntya-dvayam antyam* which means 'The Ultimate and Twice the Penultimate'. The ultimate means the last figure, and the penultimate is the figure before the last one.

Example 3

So in the number **12384** the Sutra tells us to add up the 4 and twice the 8. This gives us 20, and since 4 goes into 20 it will also go exactly into 12384.

So when using *"The Ultimate and Twice the Penultimate"* we add the last figure to twice the one before it, and if 4 divides into the result then the number is divisible by 4.
Otherwise it is not divisible by 4.

Example 4

In the number 5574 the Sutra gives us 4 plus twice 7, which is 18. But 4 will not divide exactly into 18 so 5574 is not divisible by 4.

Exercise 5

Look again at Exercise 4 and use "The Ultimate and Twice the Penultimate" to test the numbers again for divisibility by 4.

Write down the totals the *sūtra* gives you and then write down whether 4 divides into the number or not.

So far we have dealt with divisibility by 2, 3, 4 and 5. What about divisibility by 6?

Divisibility By 6

Any number which is divisible by 6 must also be divisible by 2 and by 3 because $6 = 2 \times 3$. This means that to test a number for divisibility by 6 it must pass the tests for both 2 and 3.

All numbers divisible by both 2 and 3 are divisible by 6.

Example 5

78 is divisible by 2 (as it ends in an even number) and also by 3 (as its digit sum is 6).
So 78 is divisible by 6.

However 454 is divisible by 2, but not by 3,
so it is not divisible by 6.

It is essential that both tests are passed.

Exercise 6

Test the following numbers for divisibility by 6:

a) 888	b) 6789	c) 3456	d) 94	e) 1234	f) 2468012
g) 1998	h) 390	i) 44444	j) 70407	k) 111222	l) 12345678

Divisibility By 15

Example 6

This is similar to divisibility by 6.

> Because 15 = 3 × 5 any number which is divisible by 3 and by 5 must be divisible by 15.

Example 7

Consider the number 345.
It must be divisible by 3 as its digit sum is 3, and it is also divisible by 5 as the last figure is 5.
So 345 is divisible by 15.

Example 8

But the number 4040 is not divisible by 15 because although it is certainly divisible by 5 it is not divisible by 3.

Again both tests must be passed.

Exercise 7

Test these numbers for divisibility by 15:

a) 78 b) 785 c) 1785 d) 88888 e) 12345 f) 543210

Here is a summary of all the divisibility tests we have had so far:

Number being tested	Test
2	Is the last figure 0 or even?
3	Is the digit sum 3, 6 or 9?
4	Is the 2-figure number on the end divisible by 4?
5	Is the last figure 0 or 5?
6	Is the number divisible by both 2 and 3?
9	Is the digit sum 9?
10	Is the last figure 0?
15	Is the number divisible by both 3 and 5?

Exercise 8

For each of the following numbers test for divisibility by 2, 3, 4, 5, 6, 9, 10,

a) 525	b) 45	c) 864	d) 57330	e) 88188	f) 181
g) 842	h) 999	i) 121314	j) 906030	k) 9876	l) 123456

m) What is the smallest 2-figure number that does not pass any of these tests?
n) What is the largest 2-figure number that does not pass any of these tests?
o) What is the smallest number that passes all of these tests?

Exercise 9

Find a 2-figure number divisible by
a) 4 if the last figure is a 7 (2 answers)
b) 4 if the sum of the figures is 5
c) 4 if the product of the figures is 8
d) 4 if the last figure is one more then the first (2)
e) 4 if the first figure is odd and the last is 6 (5)
f) 4 if the figures differ by 3 (3)
g) 6 if the first figure is 4 (2)
h) 6 if the first figure is greater than 7 and is odd (2)
i) 6 if the product of the figures is 56
j) 6 if the first figure is twice the last (2)
k) 6 if the last figure is 4 more than the first
1) 6 if twice the product of the figures is 36

Exercise 10

Find a 3-figure number divisible by
a) 9 if the first and last figures are both 1
b) 9 if the last figure is 6 more than the first (3)
c) 9 if the first two figures are the same and both even (4)
d) 9 if the first figure is 5 and the others are both odd (2)
e) 9 if all three figures are the same (3)
f) 9 and nearest to 200
g) 15 if the first figure is 1 and all the figures are odd (2)
h) 15 if the first figure is 8 and the others are odd
i) 15 and is nearest to 700
j) 15 which is the same when written backwards (3)
k) 15 which is less than 200 and is also an even number (3)
1) 15 which begins with an even number and contains exactly two odd figures (?)
m) 15 with all its figures odd
n) Find a 4-figure number divisible by 6 in which the figures are consecutive and
 i) ascending
 ii) descending
o) Find the largest odd 3-figure number in which the figures add up to 7
p) Find the smallest 3-figure number divisible by 5 in which the figures add up to 10

Exercise 11
Ask your teacher for more exercises.

18 Some specific multiplication and division

Multiplication By 11

1) Study the following pattern of numbers, check that each line is correct, and add three more rows:

$$\begin{aligned} 0 \times 9 &= 1 - 1 \\ 1 \times 9 &= 11 - 2 \\ 12 \times 9 &= 111 - 3 \\ 123 \times 9 &= 1111 - 4 \end{aligned}$$

Example 1

Find:

a) 52×11 and b) 57×11.

The 11 times table is easy to remember and multiplying longer numbers by 11 is also easy.

a) To multiply a 2-figure number, like 52, by 11 we write down the number being multiplied, but put the total of the figures between the two figures: 572.
So $\underline{52 \times 11 = 572}$. between the 5 and 2 we put 7, which is 5 + 2.

b) For 57×11 there is a carry because 5 + 7 = 12 which is a 2-figure number. So we get $5_1 27 = \underline{627}$ the 1 in the 12 is carried over to the 5 to give 6.

This is easy to understand because if we want, say, 52×11 we want eleven 52's. This means we want ten 52's and one 52 or 520 + 52:

$$\begin{array}{rrrl} 5 & 2 & 0 & \\ & 5 & 2 & + \\ \hline 5 & 7 & 2 & \\ \hline \end{array}$$

note how the 2 and the 5 get added in the middle column

Exercise 1

Multiply the following by 11:

a) 23 b) 61 c) 44 d) 16 e) 36 f) 50 g) 76

h) 88 i) 73 j) 65 k) 75 l) 99 m) Is 473 divisible by 11?

Example 2

Find 234×11.
3-figure numbers like this are also easy to multiply by 11.
$234 \times 11 = \underline{2574}$ the first and last figures are the same as in 234.
For the 5 in the answer we add up 2 and 3 and for the 7 we add up 3 and 4.

Example 3

Find 777 × 11.

The method above gives: $7_1 4_1 47$ = 8547. We simply carry the 1 's over.

Exercise 2

Multiply by 11:

a) 345 b) 444 c) 135 d) 531 e) 888 f) 372 g) 629

● How could this method be extended for multiplying numbers like 2345?

Exercise 3

Ask your teacher for more exercises.

Division by 9

As we have seen before, the number 9 is special and there is a very easy way to divide by 9.

Example 4

Find 23 ÷ 9.

The first figure of 23 is the answer: 2.
And we add the figures of 23 to get the remainder: 2 + 3 = 5.
So 23 ÷ 9 = 2 remainder 5.

It is easy to see why this works because every 10 contains a 9 with 1 left over.
So 2 tens contains 2 9's with 2 left over.
And if 20 contains 2 9's remainder 2, then 23 (which is 3 more) contains 2 9's remainder 5.

Exercise 4

Divide by 9:

a) 51 b) 34 c) 17 c) 17 d) 44 e) 60 f) 71 g) 46

This can be extended to the division of longer numbers.

Example 5

Find 2301 ÷ 9.

If the sum was written down it would look like this:

```
9) 2  3  0     1
   ------------
   2  5  5  R  6
```

The initial 2 is brought straight down into the answer:

```
9) 2  3  0  1
   ↓
   ----------
   2
```

this 2 is then added to the 3 in 2301, and 5 is put down:

```
9) 2  3  0  1
    ↗
   ----------
   2  5
```

this 5 is then added to the 0 in 2301, and 5 is put down:

```
9) 2  3  0  1
       ↗
   ----------
   2  5  5
```

this 5 is then added to 1 to give the remainder, 6:

```
9) 2  3  0  1
          ↗
   ----------
   2  5  5  6
```

> The first figure of the number being divided is the first figure of the answer,
> and each figure in the answer is added to the next figure in the number being divided to give the next figure of the answer.
> The last number we write down is the remainder.

Example 6

Find 1234 ÷ 9.

```
9) 1  2  3     4
   -------------
   1  3  6  R 10
```

In this example we get a remainder of 10, and since this contains another 9 we add 1 to 136 and get 137 remainder 1.

Exercise 5

Divide the following numbers by 9:

a) 212 b) 3102 c) 11202 d) 31 e) 53 f) 203010 g) 70 h) 114

i) 20002 j) 311101 k) 46 l) 234 m) 56 n) 444 o) 713

Example 7

3172 ÷ 9

9)	3	1	7		2
	3	4	11	R	13

Here we find we get an 11 and a 13: the first 1 in the 11 must be carried over to the 4, giving 351, and there is also another 1 in the remainder so we get 352 remainder 4.

Exercise 6

Divide the following by 9:

a) 6153 b) 3282 c) 555 d) 8252 e) 661 f) 4741 g) 12345 h) 4747

Exercise 7

Ask your teacher for more exercises.

19 Powers of ten and decimals

You will already be familiar with decimals: numbers like 1234.5 which have a decimal point in them. You will know that the 1 in this number represents 1000 as it is the thousands position (the fourth position before the decimal point)

Similarly the 2 means 200, the 3 means 30 and the 4 means just 4.

> The decimal point shows where the whole number part of the number ends. Past the decimal point we go into fractions: first tenths then hundredths, thousandths and so on.

... hundreds tens units **tenths hundredths thousandths ten thousandths hundred thousandths** …

1) What would you say comes after the hundred thousandths?

So the 5 in 1234.5 represents 5 tenths.

Every whole number has a decimal point, even though it may not be written.
So in the number 20 for example there is a decimal point at the end (20.) but it is not written as there is no need to write it.

Adding And Subtracting Decimal Numbers

Since we should always add like things, and not unlike things:

> When adding and subtracting decimal numbers we should always line up the decimal points, then all the columns will be lined up: tens under tens, units under units and so on.

Example 1

Find a) 45.67 + 123.3 b) 1.045 + 33 + 0.3

a)
$$\begin{array}{r@{.}l} 45 & 67 \\ 123 & 3 \quad + \\ \hline 168 & 97 \\ \hline \end{array}$$

b)
$$\begin{array}{r@{.}l} 1 & 045 \\ 33 & \\ 0 & 3 \quad + \\ \hline 34 & 345 \\ \hline \end{array}$$

The decimal points are simply lined up, remembering that in the number 33 the point is at the end.

Example 2

Find a) 76 – 1.23 b) 5 – 0.005

a)
```
7 6 . 0 0
  1 . 2 3 –
-----------
7 4 . 7 7
```

b)
```
5 . 0 0 0
0 . 0 0 5 –
-----------
4 . 9 9 5
```

You may prefer to put the point at the end of whole numbers when subtracting, and as many zeros as necessary.

> The Digit Sum Check can be used in all our decimal work, just as in sums without decimal points.

Exercise 1

Do the following decimal additions and subtractions:

a) 43.22 + 7.7	b) 103.44 + 18	c) 1.006 + 321.4	d) 87 + 1.23	e) 1.9 + 11.1+4
f) 44.44 + 5.05 + 1	g) 109.8 + 77.88	h) 17.1 + 1.71 + 0.07	i) 44.55 – 2.3	j) 43.2 – 7.15
k) 91.19 – 3.3	l) 133-1.4	m) 67.8 – 19	n) 0.66 – 0.05	o) 1.111 – 0.08

Multiplying And Dividing Decimal Numbers

This is similar to ordinary multiplication and division.

> We simply insert the decimal point when we come to it.

Example 3

Find a) 23.45 × 6 b) 23.45 ÷ 2

a) $\begin{array}{r} 2\;3\;.\;4\;5 \\ 6 \times \\ \hline 1\;4\;{}_{2}0\;.\;{}_{2}7\;{}_{3}0 \\ \hline \end{array}$

We multiply 6 by 5 and then by 4, as usual, then put the point down and continue. The final answer can be given as 140.7 since the final 0 serves no purpose.

b) $2)\;\begin{array}{l} 2\;3\;.\;{}_{1}4\;5\;{}_{1}0 \\ \hline 1\;1\;.\;7\;2\;5 \\ \hline \end{array}$

We divide from the left, putting the point down when we come to it.
The extra 0 at the end of 23.45 has been put on so that the division can be completed.

Exercise 2

Multiply/divide the following:

a) 3.45 × 3 b) 19.2 × 4 c) 0.005 × 7 d) 0.08 × 5 e) 0.6 × 7 f) 0.09 × 9

g) 3.45 ÷ 3 h) 111.6 ÷ 9 i) 123.45 ÷ 5 j) 1.2 ÷ 6 k) 8.4 ÷ 7 l) 3.6 ÷ 5

According to Vedic mathematical calculations, the following enumeration system of powers of ten is used:

10^0 *eka* (units)
10^1 *daśa* (tens)
10^2 *śata* (hundreds)
10^3 *sahasra* (thousand)
10^4 *ayuta* is ten thousand
10^5 *lakṣa* is hundred thousand
10^6 *niyuta* or one million is ten times *lakṣa*
10^7 *koṭi* is ten times *niyuta*
10^8 *arbuda* is ten times *koṭi*
10^9 *vṛnda* or one billion is ten times *arbuda*
10^{10} *kharva* is ten times *vṛnda*
10^{11} *nikharva* is ten times *kharva*
10^{12} *śaṅkha* or one trillion is ten times *nikharva*
10^{13} *padma* is ten times *śaṅkha*
10^{14} *sāgara* is ten times *padma*
10^{15} *antya* or one quadrillion is ten times *sāgara*
10^{16} *madhya* is ten times *antya*
10^{17} *parārdha* is ten times *madhya*

Multiplying And Dividing By Powers Of Ten

Take any number, say 3, and multiply it by 10.
The answer is 30.
Now divide 30 by 10.
The answer is 3. We come back to the original number, 3.

Example 4

So 55 × 10 = 550, 8760 × 10 = 87600,
and 390 ÷ 10 = 39, 7000 ÷ 10 = 700.

Any whole number can be multiplied by 10 by simply adding a 0 at the right-hand end.
And any whole number that ends in 0 can be divided by 10 by simply removing that 0.

Since multiplying by 10 means adding a 0, multiplying by 10 twice means adding two 0's. So to multiply by 100 (10 × 10) we put two 0's at the end.
Multiplying by 1000 we put three 0's, and so on.
Similarly for division: dividing by 100 involves removing two 0's, dividing by 1000 means removing three 0's, and so on.

Example 5

So 66 × 100 = <u>6600</u>, 230 × 1000 = <u>230000</u>,
and 23400 ÷100 = <u>234</u>, 9000000 ÷ 10000 = <u>900</u>.

These sums are so easy that they come under the Vedic *sūtra vilokanam* which means "By Mere Observation".

Exercise 3

Multiply/divide the following:

a) 73 × 100 b) 300 × 10 c) 16 × 1000 d) 470 × 10000
e) 500 ÷10 f) 91000 ÷100 g) 3000 ÷1000 h) 3003000 ÷100

Now we can bring in the "Proportionately" *sūtra*.

If we want to multiply a number by 20, which is 2 × 10, we need to double the number and also put a 0 on the end.

Multiplying by 300, we would multiply by 3 and put two 0's on the end.

Example 6

22 × 20 = <u>440</u>, 300 × 22 = <u>6600</u>, 80 × 40 = <u>3200</u>.

With a sum like 80 × 40 above it is best to ignore the 0's, multiply 8 by 4 to get 32, and then put 2 0's on the 32, because there are 2 0's in the sum.

Example 7

So 60 × 30 = <u>1800</u>, 300 × 20 = <u>6000</u> 90 × 4000 = <u>360000</u>.

Exercise 4

Multiply the following:

a) 333 × 20 b) 45 × 20 c) 200 × 44 d) 26 × 2000 e) 60 × 200 f) 40 × 30

g) 800 × 90 h) 460 × 200 i) 130 × 30 j) 70 × 70 k) 9000 × 9000 l) 2320 × 300

m) how many days in 400 weeks? n) how many minutes in 30 hours?

o) how many hours in 20 days?

p) if there are 30 weeks of school in a year, how many weeks of school are there in 12 years?

Now let us look at dividing by numbers like 20, 300 etc..? For dividing by 20 we need to divide by 2 and by 10.
So this means halving the number and taking a 0 from the end.

Example 8

a) 660 ÷ 20 = 33, b) 66000 ÷ 200 = 330 (we take two 0's from the 66000 here),

c) 9000 + 30 = 300, d) 2460000 ÷ 200 = 12300.

Exercise 5

Divide the following:
a) 6000 ÷ 20 b) 8000 ÷ 400 c) 1200 ÷ 60 d) 210 ÷ 70

e) 63000 ÷ 700 f) 8800 ÷ 200 g) 60 ÷ 20 h) 963000 ÷ 3000

Multiplying and dividing decimals by 10, 100 etc.

When we multiply or divide a whole number by 10, 100 etc. we are just adding or removing 0's, so we are effectively just moving the decimal point.

> Multiplying by 10, 100 , 1000 etc. means moving the point 1,2,3 etc. places to the **right.**
> Dividing by 10, 100, 1000 etc. means moving the point 1, 2, 3 etc. places to the left.

Example 9

a) 55.55 × 10 = 555.5 b) 1.234 × 100 = 123.4 c) 0.056 × 100 = 5.6,

Example 10

a) 3.69 × 100 = 369 b) 3.69 × 1000 = 3690 c) 0.7 × 100 = 70,

Here we see that if the point is moved to the end of the number we do not need to write the point. And if the point has to go further than the end of the number a zero should be added or as many zeros as are needed to move the point the required number of places.

Example 11

a) 2.3 × 20 = 46 b) 0.8 × 300 = 240.

Here we must move the point and also multiply the number.

Exercise 6

Multiply the following:

a) 55.56×10 b) 12.345×10 c) 3.7×30 d) 7.177×100

e) 0.88×10 f) 1.232×2000 g) 6.7×10 h) 22.333×300

i) 0.9×10 j) 0.9×100 k) 0.1×1000 l) 8.76×100

m) 1.3×700 n) 4.44×10000 o) 0.005×1000 p) 0.07×30000

Example 12

a) $55.55 \div 10 = \underline{5.555}$ b) $1.23 \div 10 = \underline{0.123}$ c) $15.51 \div 100 = \underline{0.1551}$,

Example 13

a) $0.123 \div 10 = \underline{0.0123}$ b) $7.7 \div 100 = \underline{0.077}$ c) $76 \div 1000 = \underline{0.076}$.

Example 14

a) $64.2 \div 20 = \underline{3.21}$ b) $18.6 \div 30 = \underline{0.62}$ c) $0.8 \div 400 = \underline{0.002}$.

Here we must move the point to the left and also divide the number (by 2, 3, 4).

Exercise 7

Divide the following:

a) $333.3 \div 10$ b) $333.3 \div 100$ c) $444.4 \div 1000$ d) $0.77 \div 10$ e) $57.9 \div 10$

f) $16.3 \div 100$ g) $8.8 \div 100$ h) $0.2 \div 1000$ i) $1234.5 \div 1000$ j) $1.2345 \div 1000$

k) $0.05 \div 10$ l) $3.04 \div 100$ m) $83 \div 100$ n) $300 \div 1000$ o) $88.88 \div 20$

p) $186.4 \div 200$ q) $90.6 \div 30$ r) $71.4 \div 70$

Exercise 8

This exercise is a mixture of the various types studied in this chapter.

a) 40×70 b) 32×100 c) 640×10 d) $1640 \div 10$ e) 20×100 f) 30×50

g) 22×40 h) 34.56×10 i) 0.5×100 j) $4000 \div 20$ k) $737.3 \div 100$ l) 300×800

m) 32×300 n) 1.23×20 o) 6.3×100 p) $6.3 \div 100$ q) 0.03×300 r) 2.002×10

s) $400 \div 10$ t) $600 \div 30$ u) $21.3 \div 30$ v) $24.68 \div 200$ w) $4.23 \div 30$ x) $2.4 \div 400$

Metric Units

Nowadays, most units we use are given using the metric system. There are certain standard units of measurement against which everything else is measured.

For example, mass is measured in kilograms. The kilogram is the mass of the International Prototype kilogram (a platinum cylinder) kept in a vault beneath the streets of Paris at the Institute of Weights and Measures.

Similarly, the meter is the distance between two specified marks on a specially shaped bar of platinum-iridium alloy, also kept at the Institute of Weights and measures in Paris. (The meter was originally determined as one 10 millionth of the distance between the equator and the pole).

Also since 1964, the liter is the standard measure for capacity (or volume of liquids).

The standard units of measurement are as follows (abbreviations are given in brackets):

Distance = meter (m)	Mass = gram (g)	Capacity = liter (l)

A convenient feature of the metric system is the fact that it uses the decimal system for larger and smaller quantities, which means to convert one unit into another, we normally only have to multiply or divide by powers of ten.

The French words 'cent', 100 and 'mille', 1000, are used for fractions of the above units. And 'kilo' is used for multiples of 1000.

The most commonly used units are:

CAPACITY
Basic unit of measurement = liter (1).
There are 1000 milliliters (ml) in a liter.

MASS
Basic unit of measurement = gram (g)
There are 1000 milligrams (mg) in a gram
There are 1000 grams in a kilogram (kg)
There are 1000 kilograms in a metric tonne (t)

LENGTH
Basic unit of measurement = meter (m)
There are 1000 millimeters (mm) in a meter (Note: There are 10 mm in a centimeter)
There are 100 centimeters (cm) in a meter
There are 1000 meters in a kilometer (km)

It is often important to be able to change one unit into another. For example, we may calculate the answer to a sum as being say 789,000 meters and it may be more concise and neater to give the answer as 789 km. Similarly, it may be better to give an answer of 0.025 kg as 25 grams.

Example 15

Convert 27,000 meters into kilometer.

There are 1000 meters in a kilometer, so we divide by 1000:
27.000 ÷ 1000 = 27km.

Example 16

Change 0.07 liters into milliliters.
There are 1000 ml in a liter, so 0.07 liters must be 0.07 × 1000 = 70 ml

Exercise 9

Change the following quantities into the required units.

a) 36,000 mm into centimeters
b) 36,000 mm into meters
c) 36,000 cm into meters
d) 234,000 m into kilometers
e) 22,000 mg into grams
f) 11,000 g into kg
g) 15,000 kg into tones
h) 36,000,000 g into tones
i) 7,560,000 ml into liters
j) 400,000 cm into km
k) 0.023 m into centimeters
l) 0.07 l into milliliters
m) 4 tones into kilograms
n) 0.33 kg into grams
o) 0.004 tones into grams
p) 0.001 g into milligrams
q) 1.5 km into meters
r) 0.0006 m into millimeters

Vedic measurements.

The following are measurements as they are mentioned in the *śāstra*.

Measurements of time (*kālaparimāṇa*).

18 nimeṣa	1 kaśtha
30 kaśtha	1 kalā (8 sec)
30 kalā	1 kṣaṇa (4 min)
6 kṣaṇa	1 daṇḍa or ghaṭikā
12 kṣaṇa	1 muhūrta
15 muhūrta	1 ahar (day)
15 muhūrta	1 rātri (night)
30 muhūrta	1 ahorātri
15 ahorātri	1 pakṣa
2 pakṣa	1 māsaḥ
6 māsaḥ	1 ayanam
2 ayanam	1 vatsaraḥ (a year)
1 vatsaraḥ	1 devadinam
360 devadinam	1 devavarṣam
12000 devavarṣam	1 caturyugam
12000 devavarṣam	4320000 manuṣyavarṣam
71 caturyugam	1 manvantaram
14 manvantaram	1 kalpam (one day of Brahma)
1 satya yugam	4800 devavarṣam
	17280000 vatsaraḥ
1 tretāyugam	3600 devavarṣam
	1296000 vatsaraḥ
1 dvāparayugam	2400 devavarṣam
	864000 vatsaraḥ
1 kaliyugam	1200 devavarṣam
	432000 vatsaraḥ

Measurement of weight (*tulāmānam*)

4 rice grain	1 gunja (2.6g)
8 gunja	1 māśa (20.6g)
2 māśa	1 pala (41.1g)
100 palas	1 tulā (4.11kg)
20 tulā	1 bhāra (82.2kg)
10 bhāra	1 ācitam (822 kg)

Measurements of length (*āṅgulimānam*)

8 yava (barley)	1 āṅgula (3/4")
10 angula	1 pradeśa (span of forefinger and thumb)
12 angulas	1 gokarṇa
18 angula	1 ratni (elbow to closed fist)
21 angula	1 aratni (elbow to small finger)
24 angula	1 hasta (elbow to tip middle finger)
2 hasta	1 gaja
3 ½ hasta	1 vyāma (space between the tips of either hand with arms fully extended)
4 hasta	1 daṇḍa
4000 hasta	1 krośa
2 krosa	1 gavyūti
4 krosa	1 yojana

Measurements of area

20 danda	1 nivarśtana
300' × 10'	1 gocarma
20 dandavargam	1 kathā
20 kathā	1 biga

Measurements of volume (*pratīmaṇam*)

8 muṣṭi (a handful)	1 kuñji
3 musti	1 nikuñjaka
4 nikuñjaka	1 kuḍava or goṣpadam
4 kudava	1 prastha
4 prastha	1 āḍhaka
4 āḍhaka	1 droṇa (a bucket)
16 drona	1 khārī
20 drona	1 kumbha
10 kumbha	1 vāha

Exercise 10
Ask your teacher for more exercises.

20 Number splitting

This is a very useful device for splitting a difficult sum into two or more easy ones.

Addition

Example 1

Suppose you are given the addition sum:

2	3	4	5	
6	7	3	8	+

With 4-figure numbers it looks rather hard.
But if you split the sum into two parts, each part can be done easily and mentally:

2	3	|	4	5	
6	7	|	3	8	+
9	0	|	8	3	

On the right we have 45 + 38 which (mentally) is 83. So we put this down. And on the left we have 23 + 67 which is 90.

Example 2

Here is one with decimals, 81.7 + 95.8:

		8	|	1	.	7
+		9	|	5	.	8
1		7	|	7	.	5

We can split the sum as shown and add 17 + 58, which is 75. Then 8 + 9 = 17. The point simply goes under the other points.

Example 3

Find 481+363.

> You will have to think where to put the line, but it is usually best to put it so that there are no carries over the line:

	4	|	8	1
+	3	|	6	3
	8	|	$_1$4	4

	4	8	|	1
+	3	6	|	3
	8	4	|	4

This example is done two in ways. The second way is easier.

Example 4

$$\begin{array}{r} 6\ 3\ 4\ 3 \\ +\ 2\ 3\ 8\ 3 \\ \hline \\ \hline \end{array}$$ It is best to insert 2 lines here: $$\begin{array}{rc|cc|c} & 6 & 3 & 4 & 3 \\ + & 2 & 3 & 8 & 3 \\ \hline & 8 & 7 & 2 & 6 \\ \hline \end{array}$$

Exercise 1

Add the following (try some of them mentally):

a) 3 4 5 6	b) 1 8 1. 9	c) 6 4 4 6	d) 8 3 2 1	e) 7 6 7	f) 3 4 5
4 7 1 7	1 7 1. 6	2 8 3 8	1 8 2 3	6 1 6	4 7 1
g) 4 4 4	h) 8 8 8	i) 5 5 1	j) 4. 5 5 4	k) 1 2 3 4	l) 5 2 3 4
2 4 6	7 0 7	6 6 2	3. 6 3 6	4 9 4 4	9 3 9 3

m) 1884 + 1908 n) 5555 + 6116 o) 9119 + 8228 p) 73839+12345

q) 2704 + 6889 r) 843 + 919 s) 7071 + 7777

Subtraction

Example 5

Consider the subtraction sum: $$\begin{array}{r} 5\ 4\ 5\ 4 \\ -\ 1\ 7\ 2\ 6 \\ \hline \\ \hline \end{array}$$

We can split this up into two easy ones: $$\begin{array}{rcc|cc} & 5 & 4 & 5 & 4 \\ - & 1 & 7 & 2 & 6 \\ \hline & & & & \\ \hline \end{array}$$

First 54 – 26, which is <u>28</u>,
then 54 – 17, which is <u>37</u>.

Example 6

$$\begin{array}{r} 4\ 4\ 6\ 8 \\ -\ 2\ 2\ 8\ 6 \\ \hline \\ \hline \end{array}$$

Splitting this in the middle like the last one would involve 68 – 86, which is not easy.
So we split as follows:

$$\begin{array}{r|rr|r} 4 & 4 & 6 & 8 \\ -\ 2 & 2 & 8 & 6 \\ \hline 2 & 1 & 8 & 2 \\ \hline \end{array}$$

Exercise 2

Subtract the following:

$$\text{a)}\ \begin{array}{r} 3\ 2\ 4\ 3 \\ 1\ 3\ 1\ 9 \\ \hline \\ \hline \end{array} \quad \text{b)}\ \begin{array}{r} 4\ 4\ 4\ 4 \\ 1\ 8\ 2\ 8 \\ \hline \\ \hline \end{array} \quad \text{c)}\ \begin{array}{r} 7\ 0\ 7\ 0 \\ 1\ 5\ 2\ 6 \\ \hline \\ \hline \end{array} \quad \text{d)}\ \begin{array}{r} 3\ 7\ 2\ 1 \\ 1\ 9\ 0\ 9 \\ \hline \\ \hline \end{array} \quad \text{e)}\ \begin{array}{r} 6\ 8\ 8\ 6 \\ 1\ 9\ 3\ 6 \\ \hline \\ \hline \end{array} \quad \text{f)}\ \begin{array}{r} 8\ 5\ 2 \\ 1\ 3\ 9 \\ \hline \\ \hline \end{array}$$

g) 66.16 – 1.61 h) 7396 – 1681 i) 5476 – 2809

Multiplication

The same technique can be applied here.

Example 7

352 × 2

We can split this sum: 35 / 2 × 2 = <u>704</u>.

35 and 2 are easy to double.

Example 8

Similarly 827 × 2 becomes 8 / 27 × 2 = <u>1654</u>,
604 × 7 becomes 6 / 04 × 7 = <u>4228</u>,
121745 × 2 becomes 12 / 17 / 45 × 2 = <u>243490</u>,
3131 × 5 becomes 3 / 13 / 1 × 5 = <u>15655</u>.

Exercise 3

Multiply the Following:

a) 432 × 3 b) 453 × 2 c) 626 × 2 d) 433 × 3 e) 308 × 6 f) 814 × 4

g) 515 × 5 h) 919 × 3 i) 1416 × 4 j) 2728 × 2 k) 3193 × 3 l) 131415 × 3

Division

Division sums can also often be simplified by this method.

Example 9

The division sum 2) 4 3 2 can be split into: 2) 4 / 32 = 216.
because 4 and 32 are both easy to halve.

Example 10

Similarly 2) 3 4 5 6 becomes 2) 34 / 56 = 1728,

6) 612 becomes 6) 6 / 12 = 102. (note the 0 here because the 12 takes up two places)
7) 2849 becomes 7) 28 / 49 = 407.
3) 24453 becomes 3) 24 / 45 / 3 = 8151.

Exercise 4

Divide the following mentally:

a) 2) 655 b) 2) 726 c) 3) 1821 d) 6) 1266 e) 4) 2048 f) 4) 2816

g) 3) 2139 h) 2) 2636 i) 2) 56342 j) 3) 91827 k) 2) 387252 l) 4) 812

Checking Devices

using The First by the First and the Last by the Last

It sometimes happens that we only want an approximate value of a calculation and so we do not need to find the exact answer. We may only want to find the first figure of the answer and the appropriate number of noughts.

The First by the First - *ādyam ādyena*

Example 11

32 × 41 is approximately 1000

by multiplying the first figure of each number together we find that 32 * 41 is approximately
30 × 40, which is 1200.
So we expect the answer to be about 1000. rounding off to the nearest thousand.

Example 12

Find the approximate value of 641 × 82.

We want the first figure of the answer and the number of 0's that come after it.
Since 600 × 80 = 48,000 and we know the answer will be more than this we can say the answer is about 50.000 (to the nearest 10,000).

Example 13

Find the approximate value of 39 × 63.

39 is close to 40 so that *"the first by the first"* gives 40 × 60 = 2400.
And we can say 2000.

Example 14

Find an approximate value for 383 × 88.

400 × 90 = 36,000 and the answer must be below this because both 400 and 90 are above the original numbers, so we can say 383 × 88 ≈ 30.000.

Note the symbol ≈ for **approximately equal to**.

Thus we see that The First by the First gives us the first figure of the answer; and the number of figures in the answer is also evident.
We may not always be certain of the first figure (as in the last example) but we will never be more than 1 out.

Exercise 5

Approximate the following:

a) 723 × 81 b) 67 × 82 c) 4133 × 572 d) 38 × 49 e) 3333 × 4444 h) 919^2

The "Last by the Last" (*āntyam antyena*)

The last figure of a calculation is also apparent by looking at the last figures in the sum.

Example 15

72 × 83 ends in 6

by multiplying the last figure of each number together we get the last figure of the answer;
2 × 3 = 6.

Example 16

383 × 887 ends in 1

since 3 × 7 = 21, which ends with a 1.

Example 17

$23 \times 48 \times 63$ ends in a <u>2</u>.

Because 3×8 ends in a 4 and 4×3 ends in a <u>2</u>.

Exercise 6

What is the last figure of the following sums?

a) 456×567 b) 76543×98 c) $67 \times 78 \times 89$ d) $789 + 987$ e) 715^2 f) 53^3 g) 888^4 h) 23.2^4

Checking Calculations

These two methods of finding an approximate value and the last figure can be used for checking the correctness of an answer.
In the following exercise some of the sums are correct and some are wrong. Can you find which is which?

Exercise 7

Which of the following sums are correct, judging by the first and last figures?

a) $627 \times 762 = 477774$ b) $715 \times 735 = 525525$ c) $54 \times 64 = 3456$ d) $84 \times 481 = 40404$

e) $593 \times 935 = 554455$ f) $592 \times 792 = 468864$ g) $726 \times 926 = 672267$ h) $462 \times 962 = 444442$

i) $741 \times 777 = 575757$ j) $408 \times 842 = 343536$ k) $733 \times 744 = 545352$ l) $37 \times 367 = 13579$

m) $223 \times 443 = 98789$ n) $538 \times 539 = 889982$ o) $265347^2 = 90409030409$

Exercise 8

Ask your teacher for more exercises.

21 All from 9 and last from 10
(*nikhilam navataś caramam daśataḥ*)

All From 9 - nikhilam navataś

Suppose we apply this part of the Sutra to a few numbers.

Example 1

Apply *All from 9* to the number 876.

We take each of the figures in this number from 9:
8 from 9 is 1, 7 from 9 is 2, 6 from 9 is 3,

so we get the number 123.

Similarly

4 9 7	7 8 6 6	6 4	2
5 0 2	2 1 3 3	3 5	7

● Check that you agree with the examples above.

● Notice that the columns all add up to 9.

Exercise 1

Apply All from 9 to the following numbers:

a) 356 b) 86413 c) 1918 d) 70891 e) 60

All from 9 and the last from 10 - nikhilam navataś caramam daśataḥ

> This is the same except that the very last figure is taken from 10 instead of 9.

Example 2

If we apply the full *sūtra* to 876 we get 124 this time, because the 6 at the end is taken from 10.

Similarly:				
3 8 8 3	6 4	9 8	6	1 0 9 0 5
6 1 1 7	3 6	0 2	4	8 9 0 9 5

Example 3
Applying the Sutra to 3450 or any number that ends in 0 we need to be a bit careful. If we take the last figure as 0 here, when we take it from 10 we get 10 which is a 2-figure number. To avoid this we take 5 as the last figure: we apply the *sūtra* to 345 and simply put the 0 on afterwards. So we get 6550.

Exercise 2

Apply All from 9 and the Last from 10 to the following:

a) 444	b) 975	c) 2468	d) 18276	e) 8998	f) 10042	g) 1020304
h) 9888	i) 52805	j) 6770	k) 9631	l) 35700	m) 1234560	n) 7

Look again at the number in part **a** above.
Your answer should be 556.
Now add the number and its 'All from 9...' number:

$$\begin{array}{r} 4\;4\;4 \\ +\;5\;5\;6 \\ \hline 1\;0\;0\;0 \end{array}$$

you get 1000.

● Try this with at least three other numbers from Exercise 2.

1) What does the number and its 'All from 9...' number add up to?

You should find that the total is always 10, 100, 1000, 10000.
The total is always one of these unities, called base numbers.
This means that when we applied the Sutra to 444 we found how much it was below 1000. Or, in other words, the Sutra gave the answer to the sum 1000 - 444. The answer is 556.

> The formula *All from 9 and the Last from 10* subtracts numbers from the next highest unity.

Example 4

1000 – 864 = 136 Just apply *All From 9 and the Last from 10* to 864,

1000 – 307 = 693,

10000 – 6523 = 3477,

100 – 76 = 24,

1000 – 580 = 420 Remember: apply the Sutra just to 58 here.

In every case here the number is being subtracted from its next highest unity.

Exercise 3

Subtract the following:

a) 1000 – 481 b) 1000 – 309 c) 1000 – 891 d) 1000 – 976 e) 100 – 78

f) 100 – 33 g) 100 – 61 h) 10000 – 8877 i) 10000 – 1357 j) 10000 – 9876

k) 1000 – 808 l) 1000 – 710 m) 1000 – 930 n) 10000 – 6300 o) 1000 – 992

In all of the above sums you may have noticed that the number of zeros in the first number is the same as the number of figures in the number being subtracted.

E.g. 1000-481 has three zeros and 481 has three figures.

First extension

Example 5

Suppose we had 1000 – 43.
This has three zeros, but 43 is only a 2-figure number.
We can solve this by writing 1000 – 043 = 957.
We put the extra zero in front of 43, and then apply the Sutra to 043.

Example 6
10000 - 58.
Here we need to add two zeros: 10000 – 0058 = 9942.

In the following exercise you will need to insert zeros, but you can do that mentally if you like.

Exercise 4

Subtract the following:

a) 1000 – 86 b) 1000 – 29 c) 1000 – 77 d) 1000 – 93 e) 1000 – 35 f) 10000 – 678

g) 10000 – 353 h) 10000 – 177 i) 10000 – 762 j) 10000 – 62 k) 10000 – 85 1) 10000 – 49

Second extension

Example 7
Now consider 600 – 77.

We have 600 instead of 100.
In fact the 77 will come off one of those six hundreds, so that 500 will be left.
So 600 – 77 – 523.
The 6 is reduced by one to 5, and the Sutra is applied to 77 to give 23.

This reduction of one illustrates the *sūtra "By One Less than the One Before"*.

Example 8

5000 – 123 = 4877 the 5 is reduced by one to 4, and the Sutra converts 123 to 877.

Example 9

Subtractions like this frequently arise when buying things.
If we offer a Rs.500 note when paying for goods costing Rs.46 the change we get should be Rs.454.

We could think of the sum as being the same as 500 – 46, so the 5 is reduced to 4, and *All from 9...* is applied to 46.

Example 10

8000 – 3222
Considering the thousands there will be 4 left in the thousands column because we are taking over 3 thousand away.

All from 9... is then applied to the 222 to give 778.
So 8000 – 3222 = 4778.

Exercise 5

Subtract the following:
a) 600 – 88 b) 400 – 83 c) 900 – 73 d) 500 – 31 e) 700 – 46 f) 200 – 69

g) 3000 – 831 h) 6000 – 762 i) 8000 – 3504j) 5000 – 1234k) 300 – 132 l) 2000 – 1444

m) 2000 – 979n) 50000 – 4334 o) 70000 – 8012 p) 6000 – 628

Combining the first and second extensions

Example 11

6000 – 32

You will see here that we have a 2-figure number to subtract from 6000 which has three zeros. We will need to apply both of our extensions together.
The sum can be written 6000 – 032.
Then 6000 – 032 = 5968.
The 6 is reduced to 5, and the Sutra converts 032 to 968.

Example 12

30000 – 63 = 30000 – 0063 = 29937 the 3 becomes 2, and 0063 becomes 9937.

Exercise 6

Subtract the following:

a) 5000 – 74 b) 8000 – 58 c) 3000 – 43 d) 7000 – 81 e) 6000 – 94 f) 4000 – 19

g) 80000 – 345 h) 30000 – 276 i) 50000 – 44 j) 700 – 8 k) 800 – 6 1) 400 – 3

m) 30000 – 54n) 20000 – 222 o) 30000 – 670 p) 70000 – 99

The final exercise is a mixture of all the types we have met:

Exercise 7

Subtract:

a) 100 – 34	b) 1000 – 474	c) 5000 – 542	d) 800 – 72	e) 1000 – 33
f) 5000 – 84	g) 700 – 58	h) 9000 – 186	i) 10000 – 4321	j) 200 – 94
k) 10000 – 358	1)400 – 81	m) 7000 – 88	n) 900 – 17	o) 30000 – 63

Exercise 8
Ask your teacher for more exercises.

22 Bar Numbers

The number 19 is very close to 20.
We can therefore conveniently write it in a different way: as $2\bar{1}$

$2\bar{1}$ means 20 – 1, and the minus is put on the top of the 1.
Similarly $3\bar{1}$ means 30 – 1 or 29.
And $4\bar{2}$ means 38.

This is like telling the time when we say 'quarter to 7' or 10 to 7' instead of 6:45 or 6:50.

We pronounce $4\bar{2}$ as 'four, bar two' because the 2 has a bar on top.

Example 1

$7\bar{2} = 68$,

$86\bar{1} = 859$, because $6\bar{1} = 59$ (the 8 is unchanged),

$127\bar{2} = 1268$, because $7\bar{2} = 68$.

Exercise 1

Convert the following numbers:

a) $6\bar{1}$ b) $8\bar{2}$ c) $3\bar{3}$ d) $9\bar{1}$ e) $7\bar{4}$ f) $9\bar{5}$ g) $5\bar{7}$ h) $46\bar{2}$ i) $85\bar{1}$ j) $774\bar{1}$

k) $999\bar{1}$ l) $1\bar{2}$ m) $11\bar{1}$ n) $12\bar{3}$ o) $3\bar{4}0$

We may also need to put numbers into bar form.

Example 2

$79 = 8\bar{1}$ because 79 is 1 less than 80

$239 = 24\bar{1}$ because $39 = 4\bar{1}$

$7689 = 769\bar{1}$ because $89 = 9\bar{1}$

$508 = 51\bar{2}$ 08 becomes $1\bar{2}$

Exercise 2

Put the following into bar form:

a) 49 b) 58 c) 77 d) 88 e) 69 f) 36 g) 17 h) 359 i) 848 j) 7719

k) 328 1) 33339 m) 609 n) 708

Example 3

How would you remove the bar number in $5\bar{1}3$?
The best way is to split the number into two parts: $5\bar{1}/3$.
Since $5\bar{1} = 49$, the answer is 493.

If a number has a bar number in it split the number after the bar.

Example 4

$7\bar{3}1 = 7\bar{3}/1 = 671$,
$52\bar{4}2 = 52\bar{4}/2 = 5162$,
$3\bar{2}15 = 3\bar{2}/15 = 2815$ since $3\bar{2} = 28$,
$5\bar{1}3\bar{2} = 5\bar{1}/3\bar{2} = 4928$ since $5\bar{1} = 49$ and $3\bar{2} = 28$,
$3\bar{1}3\bar{2}3\bar{3} = 3\bar{1}/3\bar{2}/3\bar{3} = 292827$.

Exercise 3

Remove the bar numbers:

a) $6\bar{1}4$ b) $4\bar{2}3$ c) $5\bar{2}5$ d) $3\bar{1}7$ e) $45\bar{2}3$ f) $23\bar{4}5$ g) $2\bar{2}2$ h) $333\bar{2}3$ i) $5\bar{1}32$ j) $2\bar{3}55$

k) $5\bar{4}4321$ l) $4\bar{1}3\bar{1}$ m) $6\bar{2}7\bar{3}$ n) $2\bar{1}1$ o) $41\bar{3}1$ p) $52\bar{3}3$ q) $7\bar{1}52$ r) $1\bar{3}15\bar{1}$

s) $9\bar{2}8\bar{3}$ t) $1\bar{3}1$ u) $13\bar{1}51$

All From 9 And The Last From 10

So far we have only had a bar on a single figure.
But we could have two or more bar numbers together.

Example 5

Remove the bar numbers in $5\overline{33}$.

The 5 means 500, and $\overline{33}$ means 33 is to be subtracted.
So $5\overline{33}$ means 500 – 33, and we have met sums like this in the last chapter.

500 – 33 = <u>467</u> because the 33 comes off one of the hundreds, so the 5 is reduced to 4.

And applying *"All from 9 and the Last from 10"* to 33 gives 67.

Example 6

Similarly $7\bar{1}\bar{4} = \underline{686}$ the 7 reduces to 6 and the *sūtra* converts 14 to 86,

$26\bar{2}\bar{1} = \underline{2579}$ 26 reduces to 25,
$7\bar{0}\bar{2} = \underline{698}$ the *sūtra* converts 02 to 98,
$50\bar{3} = \underline{497}$ 50 is reduced to 49 (alternatively, write $50\bar{3}$ as $5\bar{0}\bar{3}$ - see previous example),
$4\bar{2}\bar{0} = 4\bar{2}0 = \underline{380}$.

Example 7

$4\bar{2}\bar{3}1$

Here we can split the number after the bar: $4\bar{2}\bar{3}/1$.
$4\bar{2}\bar{3}$ changes to 377, and we just put the 1 on the end: $4\bar{2}\bar{3}1 = \underline{3771}$.

Example 8

Similarly $5\bar{1}\bar{2}4 = 5\bar{1}\bar{2}/4 = \underline{4884}$.

$3\bar{1}\bar{1}33 = 3\bar{1}\bar{1}/33 = \underline{28933}$,

$5\bar{1}\bar{2}\bar{3} = \underline{4877}$,

$3\bar{1}4\bar{3}\bar{1} = 3\bar{1}/4\bar{3}\bar{1} = \underline{29369}$.

Exercise 4

Remove the bar numbers:

a) $6\bar{1}\bar{2}$ b) $7\bar{3}\bar{3}$ c) $2\bar{3}\bar{1}$ d) $5\bar{1}\bar{1}$ e) $9\bar{0}\bar{4}$ f) $70\bar{6}$ g) $55\bar{2}\bar{3}$ h) $72\bar{4}\bar{1}$

i) $333\bar{2}\bar{2}$ j) $6\bar{2}\bar{1}4$ k) $5\bar{3}\bar{1}22$ 1) $33\bar{2}\bar{2}44$ m) $7\bar{3}\bar{3}\bar{3}$ n) $6\bar{1}\bar{2}\bar{3}$

o) $8\bar{3}\bar{3}\bar{2}4$ p) $44\bar{1}\bar{1}\bar{2}$ q) $74\bar{0}\bar{3}\bar{1}$ r) $7\bar{1}\bar{0}31$ s) $6\bar{3}\bar{3}\bar{2}\bar{2}$ t) $3\bar{1}10\bar{2}$

u) $4\bar{2}\bar{2}\bar{2}3$ v) $3\bar{1}\bar{1}4\bar{1}$ w) $3\bar{2}1\bar{2}\bar{2}$ x) $310\bar{2}\bar{3}$ y) $37\bar{2}\bar{0}$ z) $400\bar{3}$

Subtraction

These bar numbers give us an alternative way of subtracting numbers.

Example 9

$$\begin{array}{r} 4\ 4\ 4 \\ -\ 2\ 8\ 6 \\ \hline \\ \hline \end{array}$$

Subtracting in each column we get 4 – 2 = 2, 4 – 8 = -4, 4 – 6 = -2. Since these negative answers can be written with a bar on top we can write:

$$\begin{array}{r} 4\ 4\ 4 \\ -\ 2\ 8\ 6 \\ \hline 2\ \bar{4}\ \bar{2} \\ \hline \end{array}$$

and $2\bar{4}\bar{2}$ is easily converted into <u>158</u>.

Example 10

Similarly

$$\begin{array}{r} 6\ 7\ 6\ 7 \\ -\ 1\ 9\ 0\ 8 \\ \hline 5\ \bar{2}\ 6\ \bar{1} \\ \hline \end{array} = \underline{4859}$$

Exercise 5

Subtract using bar numbers:

a) 5 4 3 − 1 6 8

b) 5 6 7 − 2 7 9

c) 8 0 4 − 3 8 8

d) 7 3 7 − 5 5 8

e) 8 2 2 − 5 7 7

f) 6 3 3 − 8 8

g) 8 0 2 4 − 5 3 3 9

h) 6 5 4 3 − 2 8 8 1

i) 7 1 0 3 − 3 9 9 1

j) 4 5 4 5 − 1 7 9 1

k) 3 2 0 4 − 2 0 8 1

l) 5 6 4 2 3 − 2 8 1 7 2

m) 3 4 5 6 7 − 3 0 9 0 9

n) 6 7 5 4 3 − 2 7 9 6 6

o) 7 4 2 7 3 − 4 7 3 3 6

Creating bar numbers

One of the main advantages of bar numbers is that we can remove high digits in a number.

For Example writing 19 as $2\bar{1}$ means we do not have to deal with the large 9.

Example 11

Remove the large digits from 287.

Here the 8 and the 7 are large (we say that 6, 7, 8, 9 are large digits).

So we write 287 as $3\overline{13}$ the 2 at the beginning is increased to 3, and the *sūtra* is applied to 87 to give 13.

You will agree that 287 is 13 below 300, which is what $3\overline{13}$ says.

Example 12

Similarly $479 = 5\bar{2}\bar{1}$,
$3888 = 4\bar{1}\bar{1}\bar{2}$,
$292 = 3\bar{1}2$,
$4884 = 5\bar{1}\bar{2}4$,
$77 = 1\bar{2}\bar{3}$ (you can think of 77 as 077),

and so on.

Exercise 6

Remove the large digits from the following:

a) 38 b) 388 c) 298 d) 378 e) 4887 f) 39876 g) 3883 h) 1782 i) 3991 j) 3822

k) 4944 1) 390 m) 299 n) 98 o) 87 p) 888 q) 996 r) 2939 s) 1849 t) 7

u) 3827 v) 191919

Exercise 7

Ask your teacher for more exercises.

23 On the flag (dhvajāṅka)

Calculating From Left To Right

It is common to do calculations starting at the right and working towards the left. This is however not always the best way.
Calculating from left to right is much easier, quicker and more useful.

The reason for this is that numbers are written and spoken from left to right.
Also in calculations we often only want the first one, two or three figures of an answer, and starting on the right we would have to do the whole sum and so do a lot of useless work.

In this chapter all the calculations will be done mentally. We will write down only the answer.

Addition

Example 1
Given the addition sum

```
 2 3
 4 5 +
 -----
 -----
```

there is no difficulty in finding the answer. From left to right the columns add up to 6 and 8.

So the answer is 68.

Example 2

But in the sum

```
 4 5
 3 8 +
 -----
 -----
```

the totals we get are 7 and 13, and 13 is a 2-figure number.
The answer is not 713: the 1 in the 13 must be carried over and added to the 7.
This gives 83 as the answer.

This is easy enough to do mentally, we add the first column and increase this by 1 if there is a carry coming over from the second column. Then we tag the last figure of the second column onto this.

Example 3

	6	6			5	5			8	4	
	2	8	+		3	5	+		5	8	+
	9	4			9	0		1	4	2	

	5	6	
	9	6	+
1	5	2	

$\widehat{8, 14} = 94$ $\widehat{8, 10} = 90$ $\widehat{13, 12} = 142$ $\widehat{14, 12} = 152$

> In every case the tens figure in the right-hand column total is carried over to the left-hand column total.

We use the curved lines to show which figures are to be combined.

Exercise 1

Add the following mentally from left to right:

a)	b)	c)	d)	e)	f)
5 6	8 8	4 5	5 4	3 9	2 7
6 7 +	3 3 +	6 7 +	6 4 +	4 9 +	5 6 +
____	____	____	____	____	____

Example 4

1	8	7	
4	4	6	+

Here the three column totals are 5, 12 and 13 so two carries are needed.

The 1 in the 12 will be carried over to the 5 making it a 6.
So when the 5 and the 12 are combined we get 62.

The 1 in the 13 is then carried over and added onto the 2 in 62, making it 63.
So combining 62 and 13 gives the answer, 633.

It is important to get the idea of doing this mentally from left to right:

First we think of 5, the first total.
Then we have $\widehat{5, 12}$ which we mentally combine into 62.
Hold this 62 in the mind, and with the third total we have $\widehat{62, 13}$ which becomes 633

Example 5

```
 7 7 7
 4 5 6 +
 ______
 ______
```

The first two columns give $1\overset{\frown}{1, 1}2$ which becomes 122.
Then with the third column we have $12\overset{\frown}{2, 1}3$ which is <u>1233</u>.

Example 6

```
 5 5 5 5
   3 1 3
   6 2 4 +
 ________
 ________
```

Starting at the left we have $\overset{\frown}{5, 1}4 = 64$.
Then $6\overset{\frown}{4, 8} = 648$ (there is no carry here as 8 is a single figure).
Finally $64\overset{\frown}{8, 1}2 = 6492$.

Exercise 2

Add the following sums mentally from left to right:

```
a) 3 6 3     b) 8 1 9     c) 7 7 7     d) 1 3 6 9     e) 4 3 7 4
   4 5 6 +      9 1 8 +      4 4 4 +      3 8 8 3 +      5 7 4 9 +
   ______       ______       ______       ________         5 5 6
   ______       ______       ______       ________       ________
                                                         ________
```

Exercise 3

In this exercise you should work with a friend.

One of you will give your friend five sums which you will speak out loud (for Example "add sixty six and seventy seven"). Your friend will add these mentally from left to right and tell you the answer. You should decide if it is right and correct any mistake.

After five sums you will change over and your friend will give you five sums. The sums are chosen to be easy to remember.

a) 55 + 77	b) 86 + 86	c) 39 + 38	d) 48 + 84	e) 71 + 82
f) 66 + 88	g) 78 + 78	h) 47 + 48	i) 57 + 75	j) 63 + 74

If you are confident about this and you are getting them right you can try the harder sums below. Five each again. Repeat the sum if necessary.

Exercise 4

Mentally add:

a) 66 + 78	b) 92 + 79	c) 555 + 888	d) 616 + 717	e) 789 + 789
f) 77 + 69	g) 83 + 68	h) 777 + 555	i) 828 + 626	j) 678 + 678

In all these sums the numbers are held in the mind (On the Flag) and built up digit by digit until the answer is complete.

Subtraction

You probably already know how to subtract numbers but in this section you will be shown a very easy method of subtracting numbers that you have probably not seen before.

Example 7

Find 35567-11828.

We set the sum out as usual:

$$\begin{array}{cccccc}3&5&5&6&7&\\1&1&8&2&8&-\\\hline 2&&&&&\\\hline\end{array}$$

Then starting on the **left** we subtract in each column.
3 – 1 = 2, but before we put 2 down we check that in the next column the top number is larger.
In this case 5 is larger than 1 so we put 2 down.

$$\begin{array}{cccccc}3&5&^{1}5&6&7&\\1&1&8&2&8&-\\\hline 2&3&&&&\\\hline\end{array}$$

In the next column we have 5 – 1 = 4, but looking in the third column we see the top number is not larger than the bottom (5 is less than 8) so instead of putting 4 down we put 3 and the other 1 is placed *On the Flag*, as shown so that the 5 becomes 15.

$$\begin{array}{cccccc}3&5&^{1}5&6&^{1}7&\\1&1&8&2&8&-\\\hline 2&3&7&3&&\\\hline\end{array}$$

So now we have 15 – 8 = 7. Checking in the next column we can put this down because 6 is greater than 2. In the fourth column we have 6 – 2 = 4, but looking at the next column (7 is smaller than 8) we put down only 3 and put the other one *On the Flag* with the 7 as shown.

$$\begin{array}{cccccc}3&5&^{1}5&6&^{1}7&\\1&1&8&2&8&-\\\hline 2&3&7&3&9&\\\hline\end{array}$$

Finally 17 – 8 = 9.

Example 8

Find 535-138.

```
 5  3  5
 1  3  8  –
 ________
 ________
```

Here we have 5 – 1 = 4 in the first column, but in the next column the figures are the same.
In such a case we must look one further column along.

```
 5  ¹3  5
 1   3  8  –
 ___________
 3
 ___________
```

And since in the third column the top number is smaller we reduce the 4 to 3

```
 5  ¹3  ¹5
 1   3   8  –
 ____________
 3   9   7
 ____________
```

Then we proceed as before:

> We subtract in each column starting on the left, but before we put an answer down we look in the next column.
> If the top is greater than the bottom we put the figure down.
> If not, we reduce the figure by 1, put that down and give the other 1 to the smaller number at the top of the next column.
> If the figures are the same we look at the next column to decide whether to reduce or not.

Exercise 5

Subtract the following from left to right. Leave some space to the right of each sum so that you can check them later.

```
a) 4 4 4     b) 6 3     c) 8 1 3     d) 6 9 5     e) 7 6 5     f) 5 0 4
   1 8 3 –      2 8 –      3 4 5 –      3 5 8 –      3 6 9 –      2 7 5 –

g) 3 4 5 6     h) 7 1 1 7     i) 5 1 6 1     j) 9 8 7 6     k) 4 6 4 3 5
     2 8 1 –      1 7 7 1 –      1 8 3 8 –      6 7 8 9 –      2 8 4 3 8 –
```

Multiplication

Example 9

Suppose we have the sum:

$$\begin{array}{r} 2\ 3\ 7 \\ 2\ \times \\ \hline \\ \hline \end{array}$$

We multiply each of the figures in 237 by 2 starting at the left.
The answers we get are **4,6,14**.

Since the 14 has two figures the 1 must be carried leftwards to the 6.
So $4, \overset{\frown}{6, 1}4 = \underline{\mathbf{474}}$.

Again we build up the answer mentally from the left: first 4, then 4,6 = 46, then $4\overset{\frown}{6, 1}4 = 474$.

Example 10

$$\begin{array}{r} 2\ 3\ 6 \\ 7\ \times \\ \hline \\ \hline \end{array}$$

First we have 14,
then $1\overset{\frown}{4, 2}1 = 161$,
then $16\overset{\frown}{1, 4}2 = \underline{1652}$.

Example 11

For 73 × 7 we get $4\overset{\frown}{9, 2}1 = \underline{511}$ (because 49 + 2 = 51)

Exercise 6

Multiply the following from left to right:

a) $\begin{array}{r} 2\ 7 \\ 3\ \times \\ \hline \\ \hline \end{array}$	b) $\begin{array}{r} 7\ 6 \\ 6\ \times \\ \hline \\ \hline \end{array}$	c) $\begin{array}{r} 2\ 6 \\ 6\ \times \\ \hline \\ \hline \end{array}$	d) $\begin{array}{r} 7\ 2 \\ 7\ \times \\ \hline \\ \hline \end{array}$	e) $\begin{array}{r} 7\ 8 \\ 9\ \times \\ \hline \\ \hline \end{array}$	f) $\begin{array}{r} 8\ 3 \\ 3\ \times \\ \hline \\ \hline \end{array}$
g) $\begin{array}{r} 6\ 4\ 2 \\ 4\ \times \\ \hline \\ \hline \end{array}$	h) $\begin{array}{r} 2\ 5\ 6 \\ 3\ \times \\ \hline \\ \hline \end{array}$	i) $\begin{array}{r} 7\ 4\ 1 \\ 3\ \times \\ \hline \\ \hline \end{array}$	j) $\begin{array}{r} 8\ 6\ 3\ 2 \\ 4\ \times \\ \hline \\ \hline \end{array}$	k) $\begin{array}{r} 5\ 4\ 3\ 2 \\ 8\ \times \\ \hline \\ \hline \end{array}$	l) $\begin{array}{r} 4\ 0\ 9\ 7 \\ 7\ \times \\ \hline \\ \hline \end{array}$

You should now practice these without seeing the sum.
Give your friend the first five sums in the following exercise, then your friend can give five to you.

Exercise 7

Multiply mentally from left to right:

a) 55×7 b) 34×8 c) 66×6 d) 62×4 e) 55×5 f) 44×4

And to finish, if you found these fairly straight forward, here are some harder sums.

Exercise 8

a) 78×6 b) 77×7 c) 444×4 d) 345×7 e) 373×7 f) 69×9

Exercise 9

Ask your teacher for more exercises.

24 Prime and composite numbers

We are familiar with prime numbers like 7, 11, 13 that have no factors except 1 and themselves.

> Numbers that are not prime are called composite numbers because they can be expressed as a product of prime factors.

Example 1

Express 6 as a product of prime numbers.

The number 6 is not a prime number. It is therefore a composite number.
We need to write 6 as a product of prime numbers.
We can say $6 = 1 \times 6$ or $6 = 2 \times 3$, but since 2 and 3 are prime numbers and 6 is not
<u>$6 = 2 \times 3$</u> is the answer.

Exercise 1

Write the following numbers as a product of primes:

a) 14 b) 33 c) 21 d) 35 e) 9

Expressing a number as a product of prime numbers shows up the inner structure of the number and this can be very useful.

<u>Factor Trees</u>

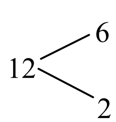
Example 2

Write 12 as a product of prime numbers.

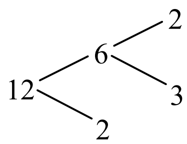
You will not find two prime numbers whose product is 12.
The answer is in fact <u>$12 = 2 \times 2 \times 3$</u>.

● Check that you agree that $2 \times 2 \times 3 = 12$.

The answer is not so obvious in this case.
One way of getting the answer is to use a Factor Tree:
Think of two numbers whose product is 12, say 6×2, and draw a sketch:



Now of 6 and 2, 6 is not prime.
But $6 = 2 \times 3$, so we extend the tree:



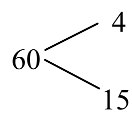
Now the numbers at the tips of the branches are all prime, so we get 12 = 2 × 2 × 3.

The order of the numbers in the answer does not matter as we get 12 whatever order we have, but it is usual to put the numbers in ascending order.

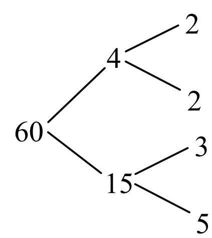
Example 3

Write 60 as a product of prime factors.

Suppose we start with 60 = 4 × 15:



Neither 4 nor 15 is prime
so we put extra branches on both:



Now we see that all the end points are prime.

So we can write <u>60 = 2 × 2 × 3 × 5</u>.

● Check that 2 × 2 × 3 × 5 gives 60.

● Starting with another product (instead of 4 × 15) which makes 60, construct a factor tree. You should still get the same answer as in the Example above.

Exercise 2

Express the following as a product of prime numbers:

a) 36 b) 28 c) 100 d) 48 e) 84 f) 32

An Alternative Method

The factor tree method can lead to rather large trees if the number is very large.
The following method avoids the need for drawing the tree.

Example 4

Write 200 as a product of prime factors.

We divide 200 by the first prime number, **2,** until it cannot be divided any more by 2, then divide in the same way by the next prime number, and so on.

So we write 200 $= 2 \times 100$,
$= 2 \times 2 \times 50$ as $100 = 2 \times 50$,
$= 2 \times 2 \times 2 \times 25$ as $50 = 2 \times 25$,
$= 2 \times 2 \times 2 \times 5 \times 5$ as $25 = 5 \times 5$.

So $\underline{200 = 2 \times 2 \times 2 \times 5 \times 5}$ which can also be written as $200 = 2^3 \times 5^2$.

Exercise 3

Express as a product of prime factors (use any method you like):

a) 4 b) 9 c) 8 d) 14 e) 16 f) 18 g) 24 h) 39 i) 42 j) 54

k) 64 1) 66 m) 72 n) 108

In fact there is an important theorem (law) in arithmetic which states that every whole number is prime or else can be written as a product of prime numbers in only one way.

So, for Example, since $12 = 2 \times 2 \times 3$ there is no different product of prime numbers that make 12: it is the only answer.

Highest Common Factor: HCF

Suppose we have two numbers: 70 and 99.

Expressing each as a product of prime factors we find: $70 = 2 \times 5 \times 7$, $99 = 3 \times 3 \times 11$.

Examining these factors we see that there is no common factor (except the number 1, which is a factor of every whole number). That is to say there is no factor of one number which is also a factor of the other number, except for 1.

Such a pair of numbers are said to be Relatively Prime, they are prime in relation to each other.

Similarly the three numbers 6, 10, 15 are relatively prime because there is no factor which divides into all three of the numbers.

• Check that you agree with this.

Now suppose we have the numbers 18 and 30.

Breaking them down into prime factors: $18 = 2 \times 3 \times 3$, $30 = 2 \times 3 \times 5$.

So 18 and 30 are not relatively prime- they have factors in common.

This means that both numbers can be divided by 2, by 3 and by $2 \times 3 = 6$.

Of these three factors the number 6 is the Highest Common Factor or HCF.

> We find the Highest common factor of two or more numbers by expressing each number as a product of prime factors, and multiplying together all the factors which the prime factors have in common.

Example 5

Find the highest common factor of 30 and 75
First find the prime factors: $30 = 2 \times 3 \times 5$, and $75 = 3 \times 5 \times 5$.
3×5 is common to both so the highest common factor is <u>15</u>.
15 is the largest number that divides into both 30 and 75.

Example 6

Find the highest common factor of 48 and 72

$48 = 2^4 \times 3$, $72 = 2^3 \times 3^2$ so the HCF is $2^3 \times 3$ which is <u>24</u>.

Example 7

Find the HCF of 24 and 75

$24 = 2^3 \times 3$, $75 = 3 \times 5^2$ therefore <u>HCF = 3</u>.

Example 8

Find the HCF for 140 and 27

$140 = 2 \times 2 \times 5 \times 7$, $27 = 3 \times 3 \times 3$ therefore <u>HCF = 1</u> and the numbers are relatively prime.

Example 9

Find the HCF for 42, 63, 90

$42=2 \times 3 \times 7$, $63 = 3 \times 3 \times 7$, $90 = 2 \times 3 \times 3 \times 5$ so <u>HCF = 3</u>.

Exercise 4

Find the highest common factor for the following numbers (you may find you can write some of the answers down after just looking at the numbers):

a) 36, 56 b) 42, 60 c) 28, 63 d) 30, 96 e) 48, 64 f) 48, 84

g) 80, 63 h) 50, 120 i) 19, 72 j) 29, 59 k) 44, 46 l) 240, 168

m) 18, 24, 60 n) 36, 42, 80 o) 30, 35, 54 p) 108, 224, 225

By Addition And By Subtraction

Another useful result, when numbers are fairly close together, is:

> The HCF will also be a factor of the sum and of the difference of the numbers.

This comes under the *sūtra* "By Addition and By Subtraction".

Example 10

Find the HCF of 411 and 417.

This means the HCF will divide into not just 411 and 417
but also 411 + 417 = 828
and 417 – 411 = 6.

This 6 is particularly useful. It tells us that the HCF is either 6 or a factor of 6 (i.e. 6 or 3 or 2 or 1).

[You can see that no number higher than 6 will divide into 411 and 417, because if, say, 7 were to divide that would mean that they were both multiples of 7 and so they would have to differ by at least 7. Similarly for any other number higher than 6]

It is easy to see that 2 is not a common factor (the last figures are not both even) and that 3 is a common factor. So <u>3 is the highest common factor</u>.

Example 11

Find the HCF of 90 and 102.

The difference here is 12, so possible common factors are 12, 6, 4, 3, 2,1, that is,
12 and all its factors.
3 is seen to be a common factor of 90 and 102,
and 2 is also, but not 4.

Therefore $2 \times 3 =$ <u>6 is the HCF</u>.

This method also indicates that if the numbers are consecutive, like say, 64, 65, then only 1 will divide into both and so they must always be relatively prime.

Exercise 5

Find the HCF:

a) 56, 58 b) 51, 54 c) 52, 55 d) 77, 79 e) 72, 78 f) 85, 100

g) 57, 69 h) 91, 98 i) 55, 75 j) 56, 66 k) 38, 49 l) 27, 28

m) 207, 213 n) 530, 534 o) 322, 343 p) 513, 531

Exercise 6
Ask your teacher for more exercises.

25 Proportionately

If we want to bake a cake or mix mortar for laying bricks we must mix the ingredients in the right proportions, or we might say, in the right ratio.

To make mortar we can mix sand and cement in the proportions 3 parts of sand to 1 part of cement. We can write this proportion, or ratio, as 3 : 1.

So you can mix 3 shovelfuls of sand with 1 shovelful of cement and (when you add the water) you will get good mortar.

Equal Ratios

But if you want a lot of mortar you may mix 6 shovels of sand with 2 shovels of cement. The ratio is still the same- you have still got 3 times as much sand as cement.

So we can write 3 : 1 = 6 : 2 - these are equal ratios.

Similarly 3 : 1 = 9 : 3 - the first number is still 3 times the second.

Which of the ratios below is the same as 3:1?

12:4 30:10 6:3 24:8 18:12 5:15

6:3 and 18:12 are not the same, as the first is not 3 times the second.

5:15 is the not same because it is the second which is 3 times the first.

12:4, 30:10 and 24:8 are equal to 3:1.

Example 1

Other Examples of equal ratios are:

8:4 = 2:1,2:3 = 4:6,3:3 = 2:2,0.5:1 = 1:2 = 5:10, 17:8 = 34:16.

Simplifying Ratios

The ratios 4:1 and 24:6 are equal but 4:1 is simpler because the numbers are small and whole. Of all the ratios equal to 4:1 the ratio 4:1 is the simplest.

Every ratio which is not a simplest ratio can be simplified to a simplest ratio.

Example 2

24:4 is equal to many other ratios, but the simplest is 6:1.
We divide the ratio by the biggest number we can.

That is to say, we divide the numbers in the ratio by their Highest Common Factor.

In this case the HCF of 24 and 4 is 4.
So we divide 24 and 4 by 4 to get 6:1.

Example 3

Similarly we can simplify the following ratios:

18:9 = 2:1 we divide by 9 here,

20:15 = 4:3 we divide by 5,

36:12 = 3:1 the HCF is 12,

100:30=10:3 the HCF is 10,

$1\frac{1}{2}:2\frac{1}{2} = 3:5$ here we multiply $1\frac{1}{2}$ and $2\frac{1}{2}$ by 2 to get whole numbers.

Example 4

If units of measurement are involved we have to be careful. A ratio of 1 minute to 1 hour is not a ratio of 1:1 but of 1:60. And a ratio of 3 days to 2 weeks is a ratio of 3:14.

Exercise 1

Write the following ratios in their simplest form:

a) 9:3 b) 5:15 c) 21:14 d) 14:10 e) 8:12 f) 8:16 g) 12:2 h) 12:3 i) 12:4

j) 12:5 k) 12:6 l) 12:12 m) 33:77 n) 60:20 o) 80:2 p) 28:16 q) 54:4 r) 200:700

s) $2\frac{1}{2}:3$ v) $4\frac{1}{2}:2\frac{1}{2}$ w) 3 weeks : 5 days z) 2 hours : 1 hour 45 minutes

Finding Equal Ratios

Instead of simplifying ratios we may, for some purposes, need to do the opposite.

Example 5

Give three ratios the same as 5:2.

We can multiply, or divide these numbers by any number we like.
So there are many answers to this question.
If we choose to multiply by 3, 4 and 5 we get 15:6, 20:8, 25:10.

2) Give three ratios the same as 2:3.
3) Give 5 ratios the same as 12:4.

Example 6

15:5 = x:6 what is the value of x?
Here we have two equal ratios, but one of the numbers is unknown.
We notice that 15 is three times bigger than 5,
so x must be three times bigger than 6.
So x = 18.

Example 7

7:3 = x:6 what is the value of x?
Here the first ratio, 7:3, is not such an easy one.
But looking at the last number in each ratio (3 and 6) we see that 6 is double 3.
So x must be double 7, which is 14, x = 14.

Example 8

x:5 = 4:10 5 is half of 10, so x = 2.

6:x = 2:7 6 is three times 2, so x = 21.

8:l = 24:xx = 3,

Exercise 2

Find x in each of the following:

a) l:3 = x:6	b) 2:10 = x:30	c) 2:3 = x:9	d) 3:7 = x:21
e) x:12 = 5:4	f) x:20 = 7:5	g) 1:x = 4:12	h) 2:x = 4:6
i) 3:2 = 15:x	j) 7:6 = 21:x	k) 4:x = 2:l	l) 50:2 = x:1

Ratio Problems – The rule of three (trairāśikam)

Many common problems are solved by using the *trairāśikam*. There are two different situations where we apply the *trairāśikam*. For the problems above we can apply the *trairāśikam* – the rule of three direct. But what about when we calculate e.g. labors working. In that process more labors are working less time the work takes. That is inversed *vyastam-trairāśikam*.

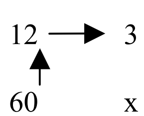
Example 9

If 12 identical pens cost Rs.60, what will be the cost of 3 pens?

This is the same as solving 12:60 = 3:x.

We may write it as following:

12 3
60 x



Now we divide 60 ÷ 12 = 5 and 5 × 3 = <u>15</u>

The solution is x = 15 so 3 pens will cost Rs.15. That is *trairāśikam*.

Exercise 3

Solve the following problems by *trairāśikam*:

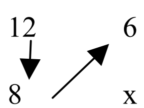
a) If 6 photos cost Rs.12, what will 4 such photos cost?
b) If 5 apples cost Rs.12, what will be the cost of 15 apples?
c) If 20 pears cost Rs.24, what will 5 cost?
d) If 4 identical bicycles cost Rs.3000, what will 20 of these cost?

Example 10

If 6 labors take 12 days to finish work in a garden, how many labors it will take to finish the work in 8 days?

This is same like solving 12 ÷ 8 = 6 ÷ x

12 6
8 x



Now we divide 12 ÷ 8 = 1 ½ and 1 ½ × 6 = <u>9</u>.

So the answer is that we need 9 labors.

Exercise 4

Solve the following problems by *vyastam-trairāśikam*:

a) If 20 labors take 12 days to carve a deity. How long it will take for 10 labors?
b) If 6 boys do a cleaning for 2 hours. How long it will take to finish it with 9 boys?
c) If 4 cooks cook a feast for 3 hours. How long it will take with 2 cooks only?
d) If 20 computers take 15 minutes to finish a calculation. How long it will take for 10 computers?

<u>Splitting in a Ratio</u>

There is another kind of problem which is solved using ratios.

Example 11
Rama and Gaura are given 20 oranges to share between them, but Rama must get 3 times as much as Gaura. What do they each get?

You have had problems like this before.

The problem could be restated as: **split 20 in the ratio 3:1**.
So two numbers add up to 20 and one is 3 times the other.

The numbers must be 15 and 5: Rama gets 15 and Gaura gets 5 oranges.

Example 12

Balarama and Vanamali share 20 laddus in the ratio 2:3. What do they each get?

You may see the answer straight away.
If not, add up the numbers in the ratio: 2 + 3 = 5.
This tells you that for every 5 laddus shared, 2 goes to Balarama and 3 goes to Vanamali.
So how many 5's of laddus are there in 20? There are four.
So Balarama gets 2 four times, and Vanamali gets 3 four times.
Then Balarama will get 8 and Vanamali will get 12 laddus.

Notice that 8 and 12 are in the ratio 2:3, and that 8 + 12 = 20 laddus.

Exercise 5

Split the following numbers in the given ratio. If the answer is obvious just write it down (and check it works), if not use the method shown in Example 11 above.

a) 15 in 2:1 b) 10 in 4:1 c) 12 in 1:3 d) 8 in 3:1 e) 18 in 5:1 f) 22 in 10:1

g) 21 in 1:2 h) 25 in 1:4 i) 15 in 2:3 j) 25 in 3:2 k) 14 in 5:2 l) 14 in 3:4

Extended Ratios

A ratio may consist of more than two numbers.

You may, for Example, see 1:2:8 or 7:3:2:5.

1:2:8 means there are three things in which the second is double the first and the last is 8 times the first.

It also shows that the last is 4 times the second (because of the ratio 2:8).

A recipe for pastry is 100g of flour, 50g of oil, 25g of water.

Of course we may want to double these amounts, or halve them etc.

But the ratio must always be 100:50:25.

Dividing this through by the HCF, that is 25, we can write 100:50:25 = 4:2:1.

In other words there must be twice as much oil as water and twice as much flour as oil.
Or if you like, 4 times as much flour as water.

Example 13

A ratio could have mixed units.

Here is a recipe for chocolate mousse:
- 2 cups of heavy cream,
- 100 grams of baking chocolate,
- 1 teaspoon of vanilla.

You could still double everything if you wanted to provided you use the right units (cups, grams, teaspoons).

Exercise 6

Simplify the following ratios:

a) 2:4:6 b) 10:2:8 c) 9:6:3 d) 8:6:4:2 e) 15:10:15 f) 12:18:24

g) 5:5:15:20 h) 4:8:8:6:2 i) 60:20:40 j) 4:3: $\frac{1}{2}$ k) $1\frac{1}{2}$:7:$2\frac{1}{2}$ 1) 1:3: $\frac{1}{4}$

m) lkg:500g:250g n) 6 cups : 8 ounces : 2 teaspoons

Example 14

Split 60 bricks in the ratio 5:3:4.

Since 5 + 3 + 4 = 12, and 60 + 12 = 5, we multiply the ratio by 5.

5 × 5 = 25, 3 × 5 = 15, 4 × 5 = 20.

So the answer is 25, 15 and 20 bricks (these can be done mentally after a little practice).

Exercise 7

Split the following numbers in the given ratio:

a) 12 in 3:2:1 b) 30 in 2:7:1 c) 24 in 3:4:5 d) 20 in 1:1:2 e) 40 in 4:1:2:3

f) 100 in 5:10:5 g) 120 in 1:2:17 h) 100 in 2:3:5 i) 72 in 3:1:3:1

Exercise 8

Ask your teacher for more exercises.

26 By one more then the one before

The Vedic Sutras give us some very fast methods of calculation.
In particular the *sūtra ekadhikena pūrvena* - By One More than the One Before gives us some special multiplication devices which are extremely efficient.

Squaring Numbers That End In 5

Squaring is multiplication in which a number is multiplied by itself: so 75 × 75 is called "75 squared" and is written 75^2.

The formula By One More Than the One Before provides a beautifully simple way of squaring numbers that end in 5.

Example 1

In the case of 75^2, we simply multiply the 7 (the number before the 5) by the next number up, 8. This gives us 56 as the first part of the answer, and the last part is simply 25 (5^2).

So 75^2 = <u>56/25</u> where $56 = 7 \times 8$, $25 = 5^2$.

Example 2

Similarly 65^2 = <u>4225</u> $42 = 6 \times 7$, $25 = 5^2$.

Example 3

And 25^2 = <u>625</u> where $6 = 2 \times 3$.

Example 4

Also since $4\frac{1}{2} = 4.5$, the same method applies to squaring numbers ending in $\frac{1}{2}$. So $4\frac{1}{2}^2 = 20\frac{1}{4}$, where $20 = 4 \times 5$ and $\frac{1}{4} = \frac{1}{2}^2$.

The method can be applied to numbers of any size:

Example 5

305^2 = <u>93025</u> where $930 = 30 \times 31$.

Even for large numbers like, say, 635, it is still easier to multiply 63 by 64 and put 25 on the end than to multiply 635 by 635.

Exercise 1

Square the following numbers:

a) 55 b) 15 c) $8\frac{1}{2}$ d) 95 e) 105 f) 195 g) 155 h) 245 i) 35 j) $20\frac{1}{2}$

k) 8005 l) 350 m) What number, when squared, gives 2025?

Multiplying numbers whose first figures are the same and last figures add up to 10,100 etc.

Example 6

Suppose we want to find 43×47 in which both numbers begin with 4 and the last figures (3 and 7) add up to 10.

The method is just the same as in the previous section.
Multiply 4 by the number *One More:* $4 \times 5 = 20$.

Then simply multiply the last figures together: $3 \times 7 = 21$.

So $\underline{43 \times 47 = 2021}$ where $20 = 4 \times 5$, $21 = 3 \times 7$.

Example 7

Similarly $\underline{62 \times 68 = 4216}$ where $42 = 6 \times 7$, $16 = 2 \times 8$.

Example 8

Find 204×206

Here both numbers start with 20, and $4 + 6 = 10$, so the method applies.

$204 \times 206 = \underline{42024}$ $(420 = 20 \times 21, 24 = 4 \times 6)$

Example 9

93×39 may not look like it comes under this particular type of sum, but remembering the *Proportionately* formula we notice that $93 = 3 \times 31$, and 31×39 does come under this type:

$31 \times 39 = 1209$ (we put 09 as we need double figures here)

so $\underline{93 \times 39 = 3627}$ (multiply 1209 by 3)

The thing to notice is that the 39 needs a 31 for the method to work here:
and then we spot that 93 is 3×31.

Example 10

Finally, consider 397×303.

Only the 3 at the beginning of each number is the same, but the rest of the numbers (97 and 03) add up to 100.

So again the method applies, but this time we must expect to have four figures on the right-hand side:

$\underline{397 \times 303 = 120291}$ where $12 = 3 \times 4$, $0291 = 97 \times 3$

Exercise 2

Multiply the following:

a) 73×77	b) 58×52	c) 81×89	d) 104×106	e) 42×48	f) 34×36
g) 93×97	h) 27×23	i) 297×293	j) 303×307	k) 64×38	l) 88×46
m) 33×74	n) 66×28	o) 36×78	p) 46×54	q) 298×202	r) 391×309
s) 795×705	t) 401×499	u) 802×499			

Rounding

There are three types of rounding:

A: rounding to the nearest 10, 100, 1000 etc.
B: significant figures.
C: decimal places.

A: Sometimes we need to round a number to the nearest 10, 100, 1000 etc.

For Example, a broadcaster may not want to report the exact number of spectators at a football match, even if the exact number is available. It may be given to the nearest thousand.

Example 11

Round to the nearest 10:

a) 4,568 b) 634 c) 625 d) 3,399

a) there is a 6 in the tens position of 4568 and since the next figure is 8 the 6 is rounded up to 7 and we write: <u>4,570</u>.

> To round to the nearest 10. Look at the figure in the tens position-
> if the figure after it is **5 or more, increase** the tens figure **by 1** and put a zero after it, if the figure after it is **less than 5, leave it as it is** and put a zero after it.

b) For 634 we see a 3 in the tens position so we leave it but replace the 4 with a 0.
So 634 rounds to <u>630</u>.

c) 625 has a 2 in the tens position but since it is followed by 5 the 2 is increased by 1 to 3: <u>630</u>.

d) For 3,399 we must increase the 9 in the tens place by 1 as it is followed by 9. So we get <u>3,400</u>.
(since $339 + 1 = 340$)

Example 12

Round a) 2,468 to the nearest 100 b) 54,321 to the nearest 1000.

a) Here we apply the same method but look at the 100's figure and the figure that follows it.
The hundreds figure of 2,468 is 4 and is followed by a 6 so the 4 is increased by 1 and we put 2 zeros at the end giving 2,500.

b) the 4 here in the thousands position in 54,321 is followed by 3 so the 4 remains as it is with 3 zeros after it: 54,000.

Exercise 3

Round the following numbers to the nearest 100:

a) 36,452 b) 73,824 d) 43,752 e) 7,629 f) 27,364

Round the following numbers to the nearest 1,000:

g) 7,428,565 h) 5,555 i) 83,652 j) 7,642,642 k) 1,234

Round the following numbers to the nearest 1,000,000:

l) 63,547,231 m) 555,456,834 n) 342,652,392 o) 13,654,372

Note: You only need to look at the appropriate figure and the next one. If you want a number to the nearest 1000, for Example, look at the 1000's and the 100's, not the 10's or units.

Significant Figures (S.F.)

A scientist needs to always state how accurate the data or results of an experiment are and will quote the number of significant figures for those numbers.

The most significant figures of a number are the figures on the left.
So if we want a number to two significant figures (often written 2 S.F.) we are interested only in the first two figures of the number, and the rest are replaced with zeros.

> We use the same rule as used in the previous section: that if the last significant figure is followed by a figure of 5 or more the last significant figure is increased by 1.

Example 13

Find 13 579 to a) 1 significant figure b) 2 significant figures
c) 3 significant figures d) 4 significant figures.

a) 1 S. F. means the first (on the left). The 1 is followed by 3 so 1 is unchanged and the other figures are replaced with zeros: 10 000.

b) 2 S.F. means the first two figures: 13. This is followed by a 5 so this 13 becomes 14, and is followed by 3 zeros: 14 000.

c) For 3 S.F, we have 135 followed by 7, so 135 becomes 136, and 2 zeros are added: 13 600.

d) For 4 S.F. we get 13 580.

Example 14

Find 0.05037 a) to 3 S.F. b) to 2 S.F.

a) 0.05037 becomes 0.0504 to 3 S.F. because 5 is the first significant figure here **- the 0's before the 5 are not counted, but the 0 within the number is counted.**

b) 0.05037 becomes 0.050 to 2 S.F. You may be tempted to leave the last 0 off and write 0.05, but you should not do this because 0.050 indicates that the number is correct to 2 S.F. whereas 0.05 looks as though it has only 1 S.F.

Decimal Places (D.P.)

This is similar to the above and is used where there are decimal places in a number. We say the number is correct to so many decimal places.

So instead of counting the figures in a number from the left-hand end as in the previous section, we count from the decimal point.

Example 15

Write the number 54.1843 correct to a) 1 decimal place b) 2 decimal places.

a) 54.1843 to 1 decimal place is 54.2. Up to one decimal place we see 54.1, but as this 1 is followed by a large figure, 8, the 1 is increased to 2.

b) 54.1843 to 2 D.P. is 54.18. The 8 in the second decimal place is followed by a small figure, 4, so we just leave it off.

Exercise 5

Ask your teacher for more exercises.

27 Algebra

Algebra is one of the main branches of the mathematics and deals with using symbols to represent things. For Example your name is a symbol which represents yourself.

In algebra we often use the letters of the alphabet as symbols to represent whatever we like. So you could say "x represents the number of children in a class" even though you don't say which class you are talking about. You, or someone else gives the symbol its meaning.

Sometimes we have to work with these symbols without knowing what they stand for.

Using Letters

Suppose we see, for Example, $2a + 3a = 5a$.

This tells us that, whatever "a" is, if 2 of them are added to 3 of them, there will be 5 of them altogether.

So $5a$ means $5 \times a$ which means $a + a + a + a + a$.

Similarly $7b - 3b = 4b$
This is like saying if there are 7 bananas and we take 3 of them away, there will be 4 bananas left.

We might have a mixture of letters in the same sum.

Example 1

$3a + 4b + 2a + 6b = \underline{5a + 10b}$

we can think that 3 apples and 4 bananas are put with 2 apples and 6 bananas, giving 5 apples and 10 bananas altogether.

When we see $3a + 4b + 2a + 6b$ we see four things added up.

We call something like $3a + 4b + 2a + 6b$ an expression, and this particular expression has four terms, since four things are added up.

Notice that $5a + 10b$ is the answer, we cannot add the 5a and the 10b together.

Another useful word is coefficient which is the number in front of a letter: so the coefficient of 5a is 5, and the coefficient of 7b is 7 and so on.

Example 2

$8a + 4a + 5b + a = \underline{13a + 5b.}$

Note here that "a" means "1a", the coefficient is 1.
So we add $8 + 4 + 1$ to get the 13a.

Example 3

$12a + 5 + 50a - 2 = \underline{62a + 3}$

here we have a mixture of "a" terms and "unit" terms:
for the "a" terms we find 12 + 50,
and for the units we find 5 – 2.

Exercise 1

Simplify the following expressions:

a) $4a + 5a$ b) $7b – 5b$ c) $7c + c$ d) $2a + 11a – 4a – a$

e) $3a + 2b + 4a + 6b$ f) $9a + 21b + a – 7a$ g) $7m +3n + 9m – 2n + 60m$

h) $8a + 5 + 4a + 9$ i) $4x + 7 – 2x – 4$ j) $a + 5b + 7 + 8b + 2a + 7$

k) $7p + 3q + p – 3q$ l) $x + y + x – y$ m) $11a + 20y – 3a – 2y + a – 8y$

We have seen above that we can only combine terms of the same kind: we can add or subtract "a" or "b" terms for Example, but we cannot combine the "a" and "b" terms together.

So if the answer is 2a + 3b we cannot add 2a and 3b because they are different types of term.

Example 4

$3x^2 + 5x^2$ means that we have x^2 three times, and also x^2 five times: so altogether we have $8x^2$, we just add the coefficients.

Example 5

Similarly in $3ab + 4bc + 5ab – bc$ we can combine the "ab" terms and the "bc" terms, because they are the same type, and the answer is $8ab + 3bc$.

Example 6

Simplify $4x^2 + 6x + 9 + 8x^2 + 5x + 1$
Here there are three different types of term: x^2, x and units. So the answer is $12x^2 + llx+ 10$.

Exercise 2

Simplify:

a) $7x^2 + 11x^2$ b) $20y^2 – 9y^2$ c) $5ab + 6ab$

d) $5ab + 8cd + 6ab – 3cd$ e) $4pq + 3qr + 6qr + 5pq$ f) $3ab + 4bc + 5cd – 2ab + bc + 7cd$

g) $xy + 3 + 8xy + 30$ h) $3x^2 + 5 + 4x^2 – 2$ i) $4a^2 + 7b^2 + 5a + 2b^2 – 4a$

j) $ax + by + ax + 2by$ k) $8x^2 + 9x + 10 + x^2 – 4x$ l) $ab + b + a + ab + 3a – b$

Brackets

Example 7

Expand 3(4a + 5b).

Since we know that 3x means x + x + x it follows that
3(4a + 5b) means 4a + 5b + 4a + 5b + 4a + 5b = 12a + 15b. So 3(4a + 5b) = 12a + 15b.
The easiest way to get the answer is to multiply 4a by 3 and also multiply 5b by 3.

Example 8

Expand and simplify 4(2x + 5) + 5(3x + 2).
Multiplying the contents of the first bracket by 4 and the second bracket by 5 we get
8x + 20 + 15x +10,
and this simplifies to 23x + 30. which is the answer.

Example 9

Find $\frac{1}{2}$(6A + 8B – 10) – 2B.

Here we halve the terms in the brackets,
so we get 3A + 4B – 5 – 2B which is 3A + 2B – 5.

Exercise 3

Simplify:

a) 5(4a + 5b)
b) 2(7x + 3y) + 4x
c) 8(5a + 5b) + 3(3a + 4b)

d) 20(7x – 9)
e) 4(17a + 23b – 70c)
f) 3(15p + 8q) + 5(p – 3q)

g) 9(a + b) + 3(a – b)
h) 20(30a + 40b + 50) + 6a
i) $\frac{1}{2}$(4x + 6y)

j) Find 3(7 + 4) by i) adding up the bracket first and then multiplying by 3,
ii) multiplying the bracket out first and then adding up.
You should get the same answer both ways.

k) Find 4(18 + 60)
l) Find 6(18 – 16) + 3(7 + 4)

Factorizing

Example 10

Fill in the brackets: 4m + 6n + 20 = 2(   ).

This is the reverse of multiplying brackets out.
The bracket, when multiplied by 2 gives 4m + 6n + 20,

so the bracket must contain 2m + 3n + 10

Example 11

Factorize 15a + 18b.

Here we first have to find the largest number that divides into both 15 and 18.
This is 3, the **highest common factor** of 15 and 18.

So we can say that 15a +18b = 3( ) and then fill in the bracket.
15a +18b = 3(5a + 6b).

This process is called **factorization**.

Example 12

Factorize 24x + 12y – 42.

The highest common factor of 24 , 12 and 42 is 6, so we write 24x + 12y – 42 = 6( ).

And then we can see that 24x + 12y – 42 = 6(4x + 2y – 7)

•Check this answer is correct by multiplying the bracket out.

Exercise 4

Copy out and fill in the brackets:

a) 8p + 12q = 2( ) b) 8p + 12q = 4( ) c) 9x – 15y = 3( )
d) 21a + 14b + 7 = 7( ) e) 54a + 36b – 18c = 6( ) f) 52x + 39y = 13( )

Factorize the following (take out the largest possible factor in each case):

g) 8a + 14b h) 9a + 12b i) 9a – 3 j) 20x + 15y k) 8x + 64

l) 4a + 6b + 8c m) 50a – 20b n) 18p + 12q o) 24a + 16b – 40c p) 18 – 9x

Example 13

Simplify and factorize 2(3a + 6b) + 3(4a – b)

Here we multiply out the brackets, then simplify and then factorize.
2(3a + 6b) + 3(4a – b) = 6a + 12b + 12a – 3b = 18a + 9b = 9(2a + b)

Exercise 5

Simplify and factorize:

a) 3(3a + 5b) + 5(a – b) b) 2(12x + 2y) + 3(3x + 6y) c) 6(2x + y) + 2(x + 3y)

d) 6(2x + 3y) + 3(x – 2y) e) 5(2x + 4y) + 2(2x + 3y) f) 2(2x + y) + 3(3x + y) + 2x

Substitution

Although it is not possible to add up something like 5a + 10b, if the values of a and b are known, the value of 5a + 10b can be found.

Example 14

If a = 3 and b = 4, find the value of 5a + 10b

If a = 3 then 5a =15, and if b = 4 then 10b = 40. So 5a + 10b =15 + 40 = 55.

The process of substitution comes under the Vedic formula *"Specific and General"* because a letter, which has no particular value, is being replaced by a particular value.

Where several calculations have to be made, as in the example below, it is important to follow the standard conventions on the order in which the calculations are done.

Example 15

If x=20 and y=3 find a) $x - y + xy$ b) $y(2x + 1)$ c) $2x^2$ d) $(2x)^2$

> The standard convention is that **brackets** are worked out first, then **multiplication and division** and finally **addition and subtraction.**

Replacing x with 20, and y with 3 (this is called substituting):

a) $x - y + xy = 20 - 3 + 20 \times 3 = 20 - 3 + 60 = 77$ note we first multiply 20 by 3 before doing the addition and subtraction.

b) $y(2x + 1) = 3(40 + 1)$ since 2x is 40 = 120 + 3 = 123 note that we work out the bracket first.

c) $2x^2 = 2 \times 20^2 = 2 \times 400 = 800$

d) $(2x)^2 = 40^2 = 1600$.

> Note carefully the difference between $2x^2$ and $(2x)^2$.

Exercise 6

Given that p = 30, q = 5 and r = 7 find:

a) $p + q + r$	b) $p - q + r$	c) $2p + 3q - 4r$	d) $3(7q - 1)$	e) $p^2 + q^2$
f) $100 - 14q$	g) $pr + qr$	h) $3pq$	i) $pqr - 1$	j) $2(p + 3q - 2r + 1)$
k) $p^2 - q^2 - r^2$	l) $2p^2$	m) $(2p)^2$	n) $(13 + r)^2$	o) $(3q^2 + 5)^2$
p) $q^2 + 8q - 7$	q) $(p + q)(r - q)$	r) $(4q - 16)(3p + 1)$	s) $(2p + 3(q - 1) + 4)$	

Multiple Substitutions

Sometimes it is useful to substitute a series of values into an algebraic expression.

Example 16

Write out the values of $2n + 1$ as n takes the values $n = 1,2,3,4$ .... etc.

When $n = 1$ $2n + 1 = 3$,
when $n = 2$ $2n + 1 = 5$,
when $n = 3$ $2n + 1 = 7$, and so on.

We see that the odd numbers are generated, starting at 3. So the answer is 3, 5, 7, 9 ....

Exercise 7

Put $n = 1,2,3,4,5$ in the following:

a) $2n$ b) $n + 5$ c) $2n – 1$ d) $3n – 2$ e) $20 – 2n$ f) $100 – 5n$

g) n^2 h) $(n + 1)^2$ i) $3n^2$ j) $(3n)^2$ k) n^3 l) $n^2 + 2n + 1$

m) $\frac{1}{2}n(n + 1)$

Note that in question a) above 2n gives the even numbers,
and that in question m) $\frac{1}{2}n(n + 1)$ gives the triangle numbers.

1) What kind of numbers are generated in c) above?

In **a** to **f** above **the coefficient of n gives the increase from one term to the next.**
In **d** for example where the coefficient of n is 3 the numbers are increasing by 3.

2) What do you think the sequence generated by **7n+3** would increase by from one term to the next?

Exercise 8

Ask your teacher for more exercises.

28 Squaring

The Vertically and Cross-wise formula simplifies nicely when the numbers being multiplied are the same, and gives us a very easy method for squaring numbers.

The Duplex

We will use the term Duplex, D, as follows:

for 1 figure D is its square, e.g. $D(4) = 4^2 = 16$;

for 2 figures D is twice their product, e.g. $D(43) = 2 \times 4 \times 3 = 24$;

Exercise 1

Find the Duplex of the following numbers:

a) 5 b) 23 c) 55 d) 2 e) 14 f) 77 g) 26 h) 90

> The square of any number is just the total of its Duplexes, combined in the way we have been using for mental multiplication.

Example 1

43^2 =1849

Working from left to right there are three duplexes in 43: D(4), D(43) and D(3).

D(4) = 16, D(43) = 24, D(3) = 9,

combining these three results in the usual way we get

16
$\overset{\frown}{16, 24}$ =184
184,9 = 1849

Example 2

64^2 = 4096

D(6) = 36, D(64) = 48, D(4) = 16,

So mentally we get

36
$\overset{\frown}{36, 48}$ = 408
$\overset{\frown}{408, 16}$ = 4096

Exercise 2

Square the following:

a) 31 b) 14 c) 41 d) 26 e) 23 f) 32 g) 2

h) 66 i) 81 j) 91 k) 56 l) 63 m) 77

Number Splitting

You may recall that we could sometimes group two figures as one when we were multiplying two 2-figure numbers together. This also applies to squaring.

Example 3

123^2 = 15129

Here we may think of 123 as 12/3, as if it were a 2-figure number:

D(12)=12^2 =144,
D(12/3) = $2 \times 12 \times 3 = 72$,
D(3) = $3^2 = 9$.

Combining these: $14\overset{\frown}{4, 7}2$ = 1512, and 1512,9 = 15129

Exercise 3

Square the following, grouping the first pair of figures together.

a) 121 b) 104 c) 203 d) 113 e) 116 f) 108 g) 111

Example 4

312^2 = 97344

Here we can split the number into 3/12 but we must work with **pairs of digits**:

D(3) = 9, D(3/12) = 72, D(12) = 144.

Combining: 9,72 = 972 we can put both figures of 72 after the 9,
$97\overset{\frown}{2, 1}44$ = 97344

Exercise 4

Square the following, grouping the last 2 figures together:

a) 211 b) 412 c) 304 d) 902 e) 407 f) 222 g) 711

Algebraic Squaring

Exactly the same method gives us squares of algebraic expressions.

Example 5

Find $(x + 5)^2$.

This is just like squaring numbers: we find the duplexes of x, x + 5 and 5.

$D(x) = x^2$, $D(x + 5) = 2 \times x \times 5 = 10x$, $D(5) = 5^2 = 25$.
So $(x + 5)^2 = x^2 + 10x + 25$.

Example 6

Find$(2x + 3)^2$.

There are three Duplexes: $D(2x) = 4x^2$, $D(2x + 3) = 2 \times 2x \times 3 = 12x$, $D(3) = 9$.

So $(2x + 3)^2 = 4x^2 + 12x + 9$.

Example 7

Find $(x – 3y)^2$.

Similarly: $D(x) = x^2$, $D(x – 3y) = 2 \times x \times -3y = -6xy$, $D(-3y) = 9y^2$.

So $(x – 3y)^2 = x^2 – 6xy + 9y^2$.

Exercise 5

Square the following:

a) (3x + 4)	b) (5y + 2)	c) (2x – 1)	d) (x + 7)
e) (x – 5)	f) (x + 2y)	g) (3x + 5y)	h) (2a + b)
i) (2x – 3y)	j) (x + y)	k) (x – y)	l) (x – 8y)

Exercise 6

Ask your teacher for more exercises.

29 Area

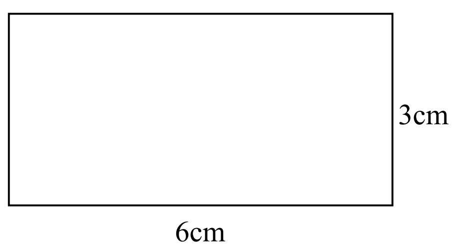
Area is the amount of space inside a shape. It is measured in square units: square kilometers, square meters, square inches, square centimeters etc.

Rectangles And Squares

For a symmetrical shape like a rectangle it is easy to find the area.

Example 1

What is the area of the rectangle below?



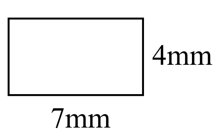
If the rectangle was filled with square centimeters (sq.cm.) there would be 6 of them in the bottom row, and since there would be 3 rows of 6 there would be 18 altogether. So the area is exactly 18 sq.cm.

We clearly find the area of such a shape by multiplying the base of the rectangle by its height: Area of a rectangle = base × height.

And this formula is true for all rectangles, even if the sizes are not whole numbers.

Example 2

Find the area of a rectangle 7mm by 4mm.



Base x height = $7 \times 4 = 28\ mm^2$ (28 square millimeters).

Exercise 1

Find the areas of the rectangles with the given sizes:

a) 7cm by 30cm b) $2\frac{1}{2}$cm by 8cm c) 2.3m by 30m

d) 48in by 42in e) 300m by 7m f) 3.5km by 8km

A square is even easier since the height is equal to the base:

Area of a square = base squared.

Example 3

Find the area of a square with a base length of 40m. Area = $40^2 = 1600m^2$.

Example 4

Find the area of a rectangle 30cm by 2m.

Here we must be careful as we must give the answer in square centimeters or square meters. Since 2m = 200cm we can give the area as 30 × 200 = 6000 sq.cm. This could also be written as 6000 cm^2.

Exercise 2

Find the areas of the rectangles with the given sizes (where different units are involved it is usually best to use the smaller unit):

a) 7cm by 2m b) 2 ½ m by 2cm c) 2m by 30cm d) 300m by 7km

e) 3.5m by 8cm f) 3 feet by 3 inches

Find the area of a square of side g) 5km h) 65cm

Irregular shapes

Finding the area of an irregular shape like a leaf is not so straightforward: we can only expect to get an approximate value for the number of square centimeters in it.

1) Place your hand on a sheet of centimeter squared paper and draw round it.
You are going to estimate the area of your hand by counting the square centimeters inside it.

Put a dot in each complete square to ensure you do not count it twice.
But what do we do with the squares which are not complete?
We count each one as a whole square if it is greater than half a square, otherwise we do not count it at all.

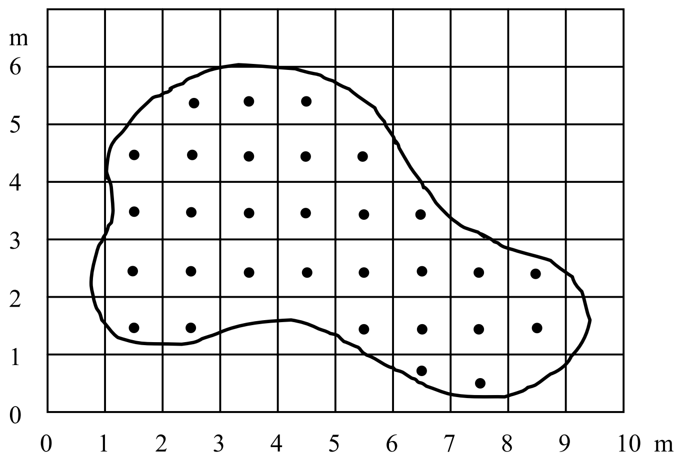
Find the area of your hand.

* Choose some other irregular shapes such as leaves and find their area in this way.

Though it is not possible to find an exact value for the area of an irregular shape like a island, it is possible to get a good estimate.

Example 5

Find the area of the garden shown.



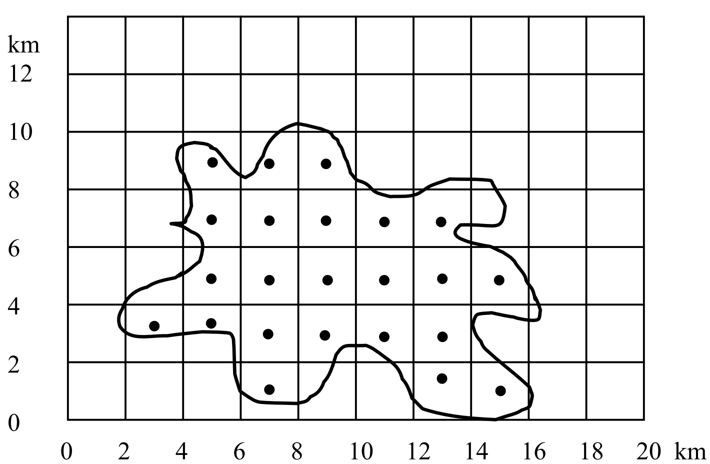
The method we use is to count up the whole squares inside the shape, and the parts which are not whole squares are counted as a whole squares if they are more than half covered by the garden.

We find 30 such squares and so the area of the garden is estimated as $30\,m^2$.

A scientist is investigating the rainfall on some remote islands and needs to know the size or area of the islands. The scientist has some maps of the region and can estimate the areas from them.

Example 6

The diagram below shows a map of an island on a grid. Estimate the area of the island.

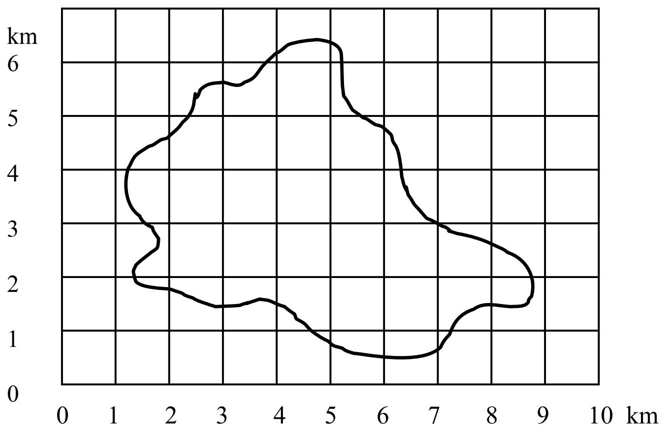


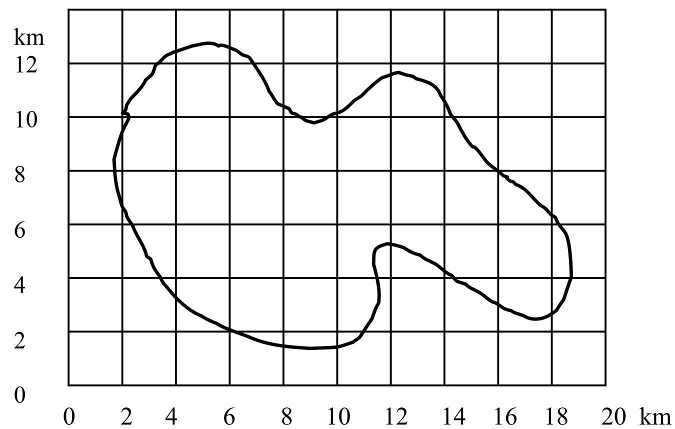
Using the same method as before we count the total number of dots to be 23.

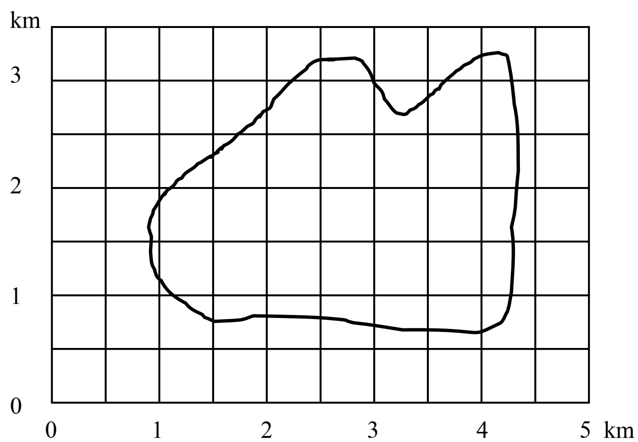
But the area is not 23 square kilometers because the grid is drawn every 2 kilometers. This means the area of each square on the grid is 4 square kilometers, and so the area of the island is approximately $4 \times 23 = 92$ square kilometers.

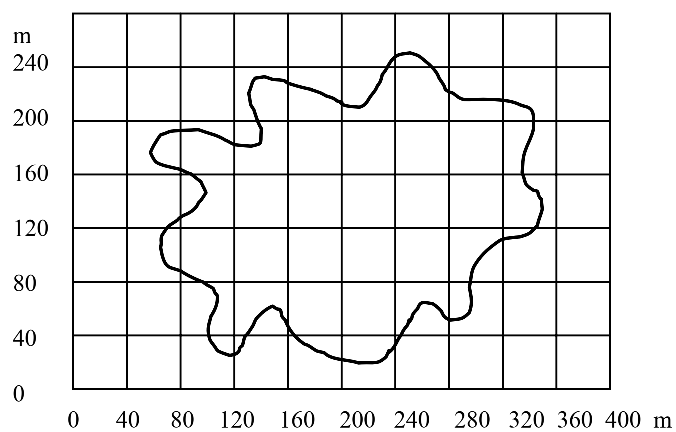
Exercise 3

Use the above method to estimate the area of the irregular objects below:







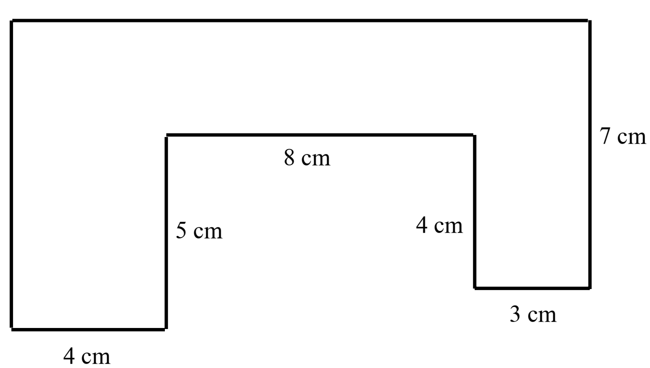


Composite Shapes

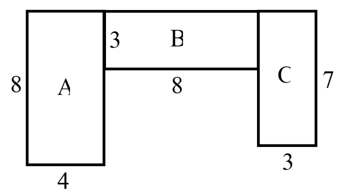
Shapes which can be broken down into rectangles can also be measured.

Example 7

Find the area of this shape:



We can extend the vertical 4cm and 5cm lines to form three rectangles A, B, C:



* Check that you agree with the figures on this diagram.

Applying the *sūtra* - *guṇita-samucchayaḥ samucchaya-guṇitaḥ* - The area of the whole is the sum of the areas we get Area = $4 \times 8 + 8 \times 3 + 3 \times 7 = 77\,\text{cm}^2$.

The above shape could also be split up by extending the horizontal 8cm line.

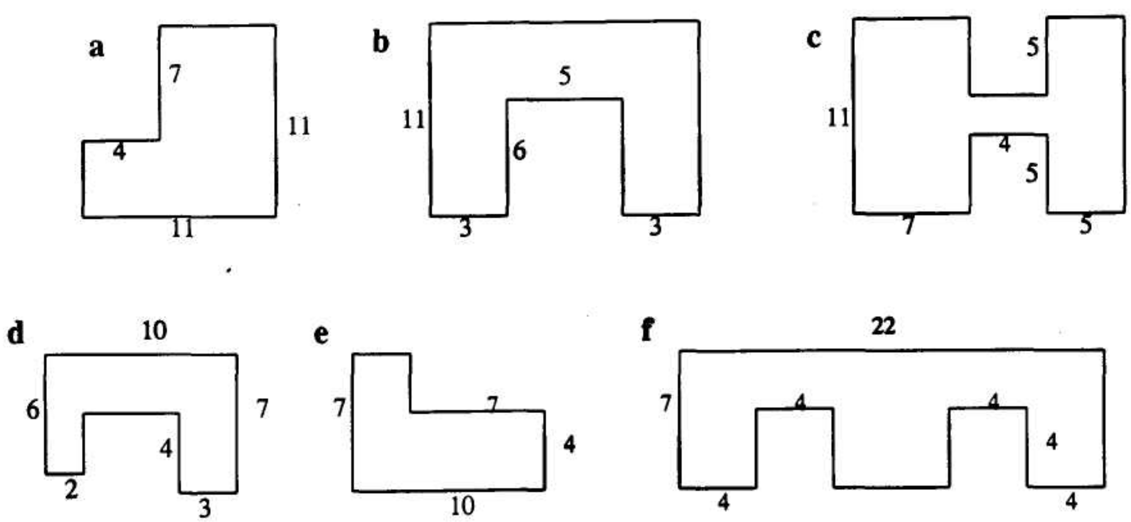
2) Show that the total area obtained in this way is also $77\,\text{cm}^2$.

It is sometimes useful to use the *sūtra* - *pūraṇāpūraṇābhyām* - in these problems.
In the above example we could enclose the figure in a rectangle, find its area, and subtract the unwanted part.

3) Find the area by this method.

Exercise 4

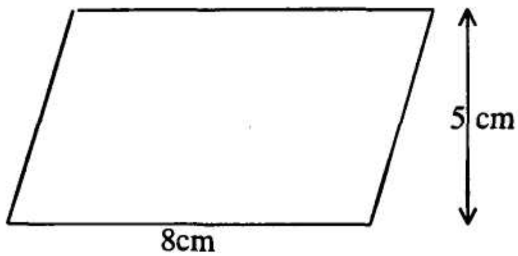
Find the areas (assume all lengths are cm):



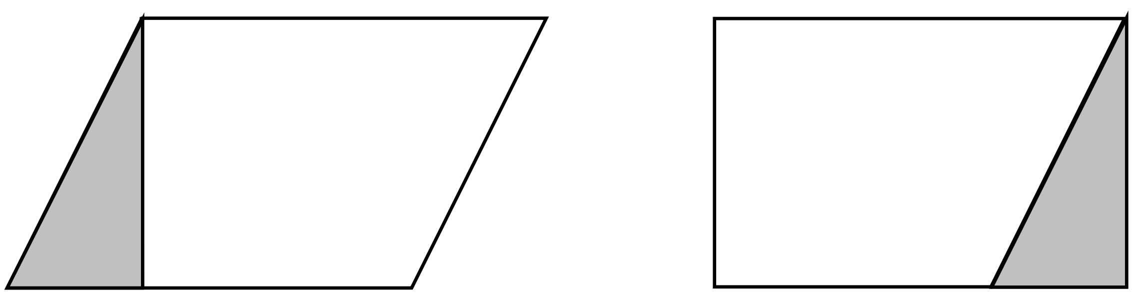
Parallelograms

Example 8

Find the area of the parallelogram below.



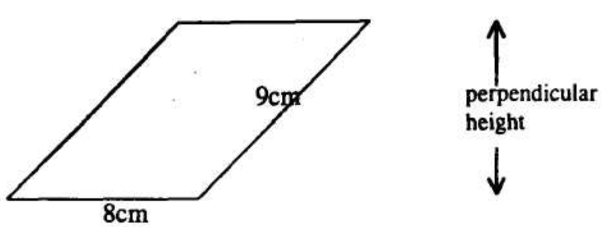
We can apply the formula "By Addition and By Subtraction" here and move the triangle on the left to the right-hand side of the figure:



Since the area of the rectangle on the right is the same as the parallelogram on the left, and since the base and height of the rectangle are 8cm and 5cm, the area of both figures is $8 \times 5 = 40\,\text{cm}^2$.

Since this can be done for any parallelogram we can use the formula:

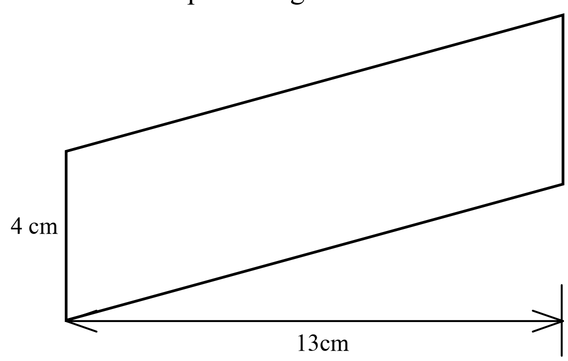
Area of a parallelogram = base x perpendicular height.
It is important to emphasize that the height is perpendicular because if we had a parallelogram like this



the area would not be 8×9. We need to know the perpendicular height, which is not given. Sometimes it is useful to think that the base of a parallelogram is not at the bottom of the figure.

Example 9

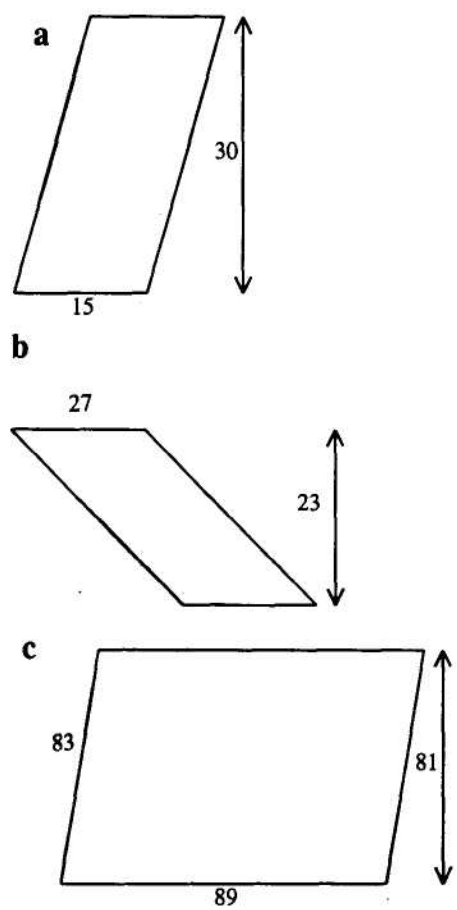
Find area of the parallelogram.

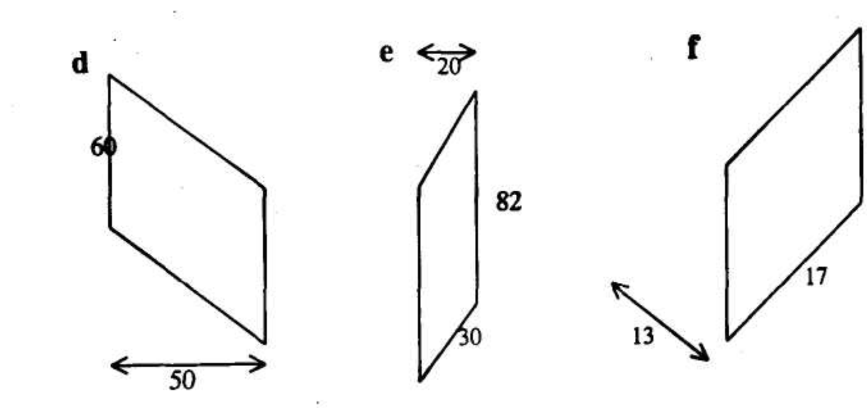


Here it is best to think of the 4cm line as the base of the parallelogram. Then 13cm will be the perpendicular height. So the area is $4 \times 13 = 52\,\text{cm}^2$.

Exercise 5

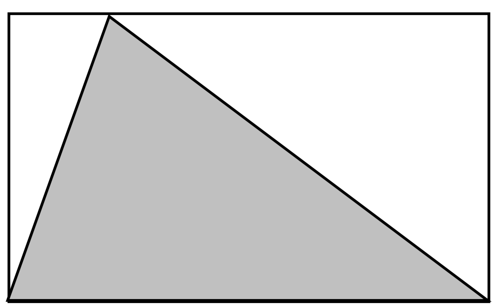
Find the areas (all units are m):



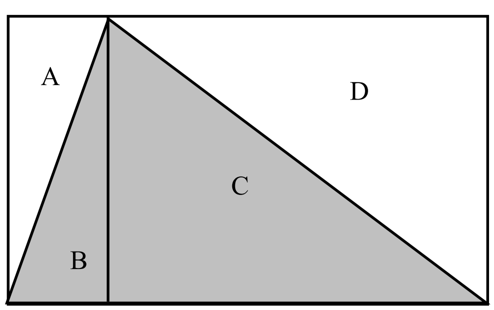


Triangles

Suppose that we have a triangle and we complete the rectangle which encloses the triangle:

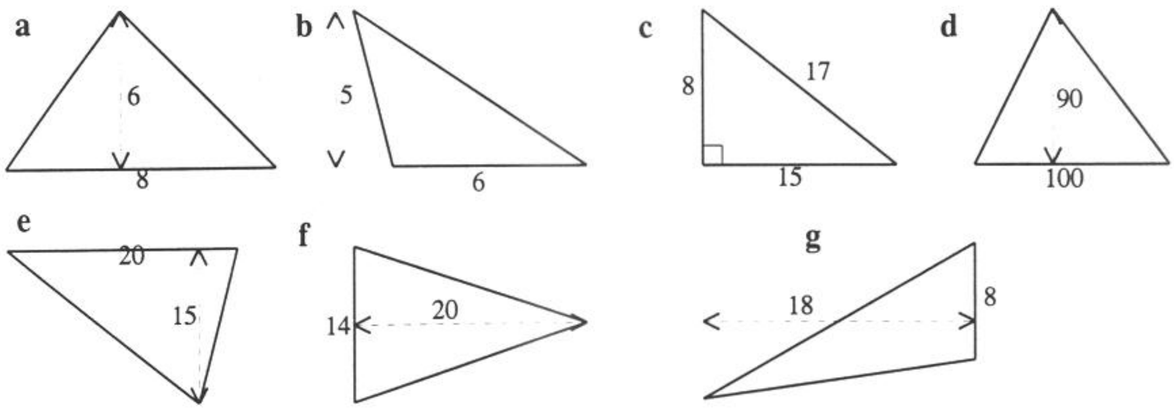


Next we drop a perpendicular from the top apex:



We can see from the diagram that the areas A and B are equal, and that the areas C and D are equal. This means that the area of the rectangle enclosing the triangle must be exactly twice the area of the triangle itself. And so:

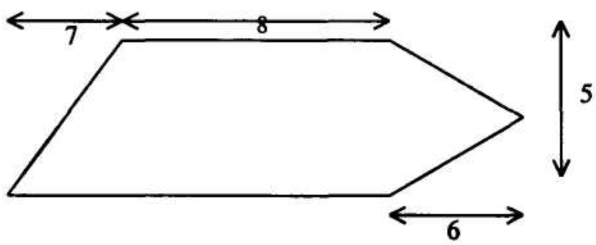
Area of a Triangle = A base × perpendicular height.



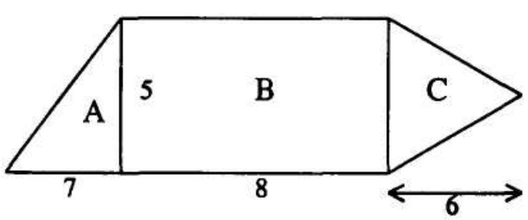
Exercise 6

Find the area of the triangles below:
This means we can now find areas of triangles and also of shapes which can be split up into triangles, rectangles or parallelograms.

Example 10



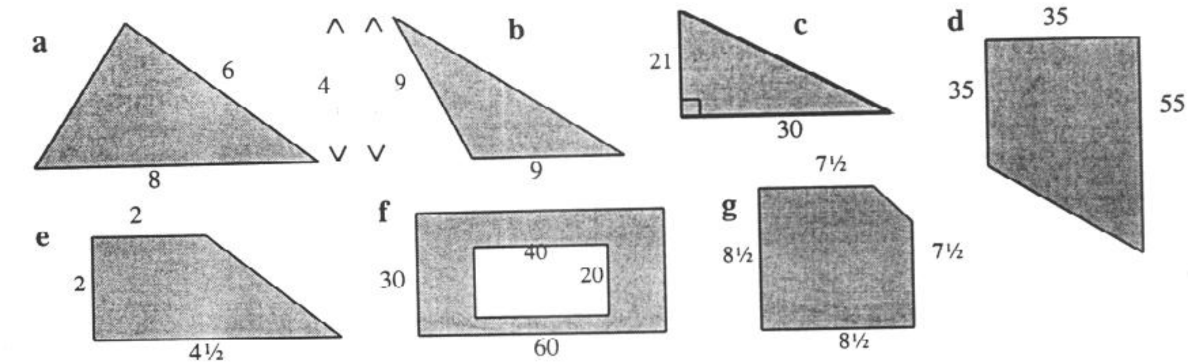
We can split this into two triangles and a rectangle:



Find the area of the figure below:
The area of A is $\frac{1}{2} \times 7 \times 5 = 17\frac{1}{2}$
the area of B is $8 \times 5 = 40$,
the area of C is $\frac{1}{2} \times 5 \times 6 = 15$. (think of the side of length 5 as the base, so that 6 is the height) And so the total area is $17\frac{1}{2} + 40 + 15 = 72\frac{1}{2}$ square units.

Exercise 7

Find the shaded areas below (all units are m):



Units Of Area

Care needs to be taken when two or more units are involved in area problems.

Example 11

Find the area of a rectangle 12mm by 1.5cm

a) in square centimeters,
b) in square millimeters.

Answers
a) The rectangle is 1.2cm by 1.5cm, so its area is $1.2 \times 1.5 = 1.8$ sq.cm.
b) The rectangle is 12mm by 15mm, so its area is $12 \times 15 = 180$ sq.mm.

Note that the 180 is 100 times bigger than 1.8.

To change square centimeters to square millimeters we multiply by 100,
because there are 100 square millimeters in a square centimeter.

Similarly there are 10,000 square centimeters in a square meter, and 144 square inches in a square foot.

Exercise 8

Find the area of the rectangle in the units stated in brackets:

a) 3m by 50cm (sq.m.) b) 3m by 50cm (sq.cm.) c) 3.2m by 0.2m (sq.m.)

d) 3.2m by 0.2m (sq.cm.) e) 2ft by 5ft (sq.in.) f) 1.2m by 1.2cm (sq.mm.)

Exercise 9

Ask your teacher for more exercises.

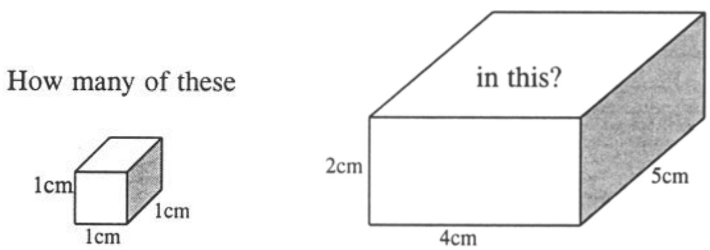
30 Volume

The volume of a solid or hollow object is the amount of space inside it.

Volume is measured in cubic units. For example it may be measured in cubic centimeters, cubic meters, cubic feet and so on.

A cubic centimeter is a cube all of whose sides are lcm in length.

Example 1



The base of the cuboid on the right here has an area of 20 sq.cm. and so 20 of the cubes on the left will fit onto it.

And since the cuboid is 2cm high there will be 2 layers of 20, making 40 cubes.

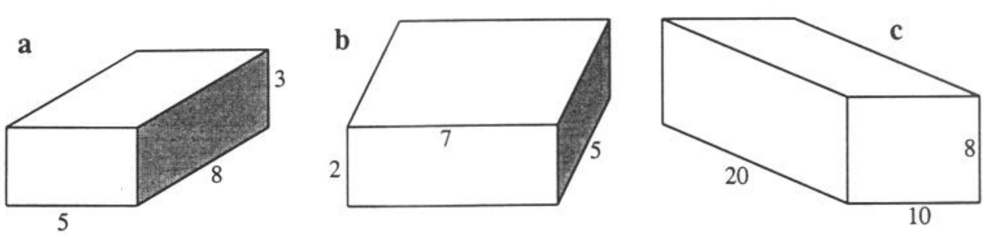
So the volume of the cuboid is 40 cubic centimeters, written $40\,cm^3$.

> The volume of a cuboid is **Length × Breadth × Height**.
> or Volume of a cuboid is **Base Area × Height**.

You will see that the answer above is found by multiplying the numbers 4, 5 and 2 together and that this will always give the volume of a cuboid.

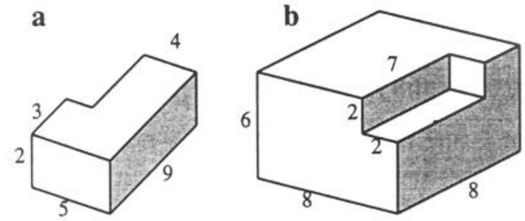
Exercise 1

Find the volume of the figures below (all units are m):



Example 2

Find the volume of these shapes all sizes being in cm.



a) The first shape could be cut into two shapes which are both cuboids (using the volume of the sum is the sum of the volumes). The height of both cuboids is 2 and so the lower volume is $5 \times 3 \times 2 = 30$, and for the upper part we get $6 \times 4 \times 2 = 48$.

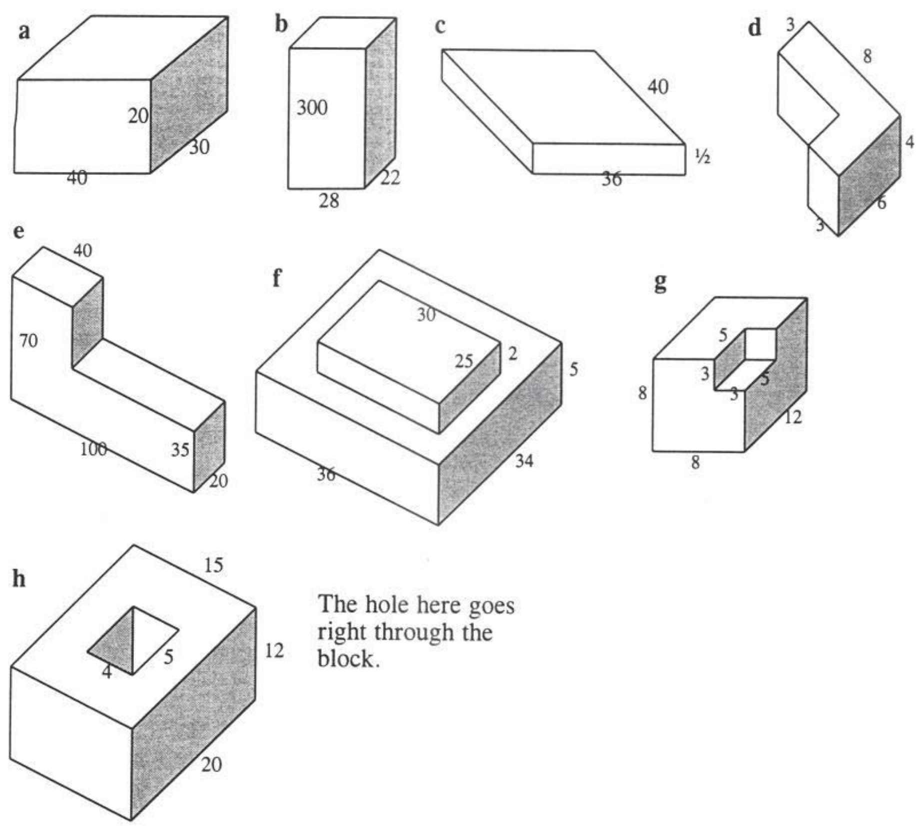
So the total volume is $30 + 48 = 78$ cm^3.

b) Here we could use By the Non-Completion and find the volume of the large cuboid and take the volume of the missing part away.

$8 \times 8 \times 6 - 2 \times 7 \times 2 = 384 - 28 = 356$ cm^3.

Exercise 2

Find the volumes of the figures below (all units are cm):



Capacity

The word capacity has the same meaning as volume but it is used where volumes of liquid are concerned.

Some units of capacity are the pint, gallon, liter and milliliter.

You should know that

8 pints = 1 gallon,
1000 milliliters = 1 liter,
1 cm^3 = 1 ml (1 ml means 1 milliliter)

Example 3

A carton of fruit juice has dimensions 10cm by 17cm by 6cm.

a) What is its volume in cm^3?

b) What is its capacity in milliliters?

c) What is its capacity in liters?

Answers

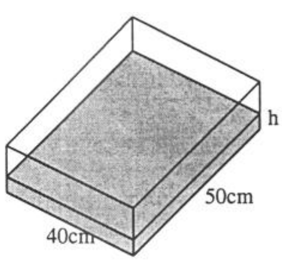
a) Treating the carton as a cuboid its volume is $10 \times 17 \times 6 = 1020$ cm^3.

b) 1020 cm^3 is the same as 1020 milliliters.

c) Since 1000 milliliters make 1 liter we divide the above answer by 1000 to convert to liters:
1020 ml =1.02 liters.

Example 4

8 liters of water are poured into a rectangular tray. If the base of the tray measures 50cm by 40cm what will be the depth of water in the tray?



8 liters of water means 8000 cm^3 and so assuming the sides of the tray are vertical the volume of the cuboid of water will be 8000 cm^3. We need the height of this cuboid. If its height is h we can say that $50 \times 40 \times h = 8000$.
This means that $2000 \times h = 8000$.

And we can see from this that h must be 4. So h = 4 cm.

Exercise 3

a) Find the capacity of a carton measuring 12cm by 5cm by 16cm
   i) in ml,
   ii) in litres.

b) An underground petrol tank has dimensions 6m by 450cm by 200cm. Find
   i) its volume,
   ii) its capacity in ml
   iii) its capacity in liters.

c) How many liters will a cuboid with dimensions 20cm, 30cm, 40cm contain?

d) How many liters will be contained in a cuboid 2m by 2m by 3m?

e) 6 liters of water is poured into a rectangular tray with base dimensions 40cm by 50cm. What will be the depth of water in the tray?

f) 160 liters of oil is poured into a rectangular container whose base is 4m by 20cm. What will be the depth of the oil?

g) Describe a method you could use to find the volume of an irregular shaped container.

Exercise 4

Ask your teacher for more exercises.

31 Fractions (*kalasa varnam*)

A fraction shows how a part is related to a whole.

Fractions are written like $\frac{4}{7}$ where the top number is called the **numerator,** and the bottom number is called the **denominator.**

And we say "four sevenths", or "four over seven".

Top-Heavy Fractions

A fraction like $\frac{7}{4}$ is **top-heavy** because the numerator is greater then the denominator. They are also called improper fractions or vulgar fractions.

Top-heavy fractions are always greater then the whole one.

In case of $\frac{7}{4}$ since $\frac{4}{4}=1$ whole, $\frac{7}{4}$ must be $1\frac{3}{4}$ (4 quarters + 3 quarters).
A number like $1\frac{3}{4}$ is called a **mixed number** because it is a mixture between a whole number and a fraction.

In fact we can always convert a top-heavy fraction into a mixed number by dividing the numerator by the denominator.

Example 1

Convert $\frac{33}{5}$ into a mixed number.
We divide 5 into 33 and get 6 remainder 3, so there are 6 whole ones.
So $\frac{33}{5} = 6\frac{3}{5}$.

Example 2

Convert $3\frac{4}{5}$ to a top-heavy fraction.
This is the reverse process, and we need to know how many fifths in $3\frac{4}{5}$.
Since each whole has 5 fifths there will be 15 fifths in 3 whole ones, and adding the 4 fifths gives 19 fifths altogether.

So $3\frac{4}{5} = \frac{19}{5}$

Example 3

Convert $7\frac{2}{3}$.
There are 21 thirds in 7 whole ones, adding 2 more gives $7\frac{2}{3} = \frac{23}{3}$.

The quick way here is to multiply 3 by 7 and add the 2 on, this gives 23 more easily. Similarly with Example 2: $5 \times 3 + 4 = 19$.

Exercise 1

Convert the following top-heavy fractions:

a) $\frac{9}{4}$ b) $\frac{17}{5}$ c) $\frac{19}{3}$ d) $\frac{7}{2}$ e) $\frac{10}{2}$ f) $\frac{36}{5}$ h) $\frac{11}{6}$ i) $\frac{60}{7}$ j) $\frac{300}{70}$ k) $\frac{39}{9}$ l) $\frac{36}{4}$

Convert the following into top-heavy fractions:

m) $3\frac{3}{4}$ n) $5\frac{2}{3}$ o) $9\frac{1}{7}$ p) $2\frac{4}{9}$ q) $1\frac{7}{10}$ r) $11\frac{2}{5}$ s) $1\frac{5}{6}$ t) $8\frac{8}{9}$ u) $20\frac{29}{30}$

Finding A Fraction Of A Number

We have already had questions like finding $\frac{3}{4}$ of something..

Example 4

Find $\frac{3}{5}$ of 20
We first find $\frac{1}{5}$ of 20 by dividing it by 5. This gives us 4.
Then if $\frac{1}{5}$ of 20 is 4, $\frac{3}{5}$ of 20 must be three times as much, which is 12.
So $\frac{3}{5}$ of 20 is 12.

We can also multiply by the numerator first, and then divide by the denominator, we get the same answer:

$\frac{3}{5}$ of 20 = $\frac{1}{5}$ of 60 = 12

Note also that since "3 of 20" means 3 20's, or 60, "3 of 20" is the same as 3×20.
So also $\frac{3}{5}$ of 20 is exactly the same as $\frac{3}{5}$ x 20.

Example 5

Find $\frac{5}{6}$ x 2.

This example shows that it is sometimes better to multiply by the numerator first.

$\frac{5}{6} \times 2 = \frac{10}{6}$

Example 6

Find $\frac{2}{3} \times 120$

Since $\frac{1}{3}$ of 120 is 40, $\frac{2}{3}$ of 120 must be 80.

Or, $\frac{2}{3} \times 120 = \underline{80}$

To find a fraction of a number we either divide the number by the denominator and then multiply by the numerator, or we multiply the number by the numerator first and then divide by the denominator.

Example 7

Find $\frac{7}{4} \times 4.4$

$4.4 \div 4 = 1.1$
$7 \times 1.1 = \underline{7.7}$

The alternative method of multiplying by the numerator first is not easier here:
we would have to multiply 4.4 by 7, and then divide by 4.

The last example shows us that we can still find a fraction of something even if the numerator is greater than the denominator. Since $\frac{7}{4} = 1\frac{3}{4}$ we have found $1\frac{3}{4}$ of 4.4.

Example 8

$1\frac{2}{5} \times 15 = \frac{7}{5} \times 15 = 7 \times 3 = \underline{21}$ (since $1\frac{2}{5} = \frac{7}{5}$).

Finding a fraction of a number comes under the “Proportionately” *sūtra*, *anurupyena*.

Exercise 2

Find (give answers cancelled and as mixed numbers where appropriate):

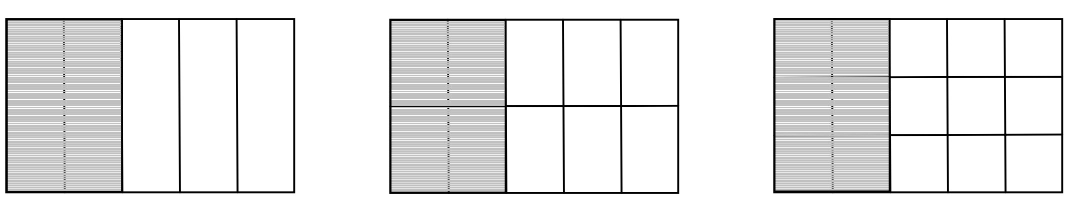
a) $\frac{2}{5}$ of 30 b) $\frac{3}{4}$ of 80 c) $\frac{4}{7} \times 2$ d) $\frac{5}{6} \times 2$ e) $\frac{7}{10} \times 7$ f) $\frac{2}{3} \times 33$

g) $\frac{7}{8} \times 40$ h) $\frac{5}{3} \times 12$ i) $\frac{9}{2} \times 40$ j) $\frac{11}{3} \times 3$ k) $\frac{3}{7} \times 140$ l) $\frac{30}{17} \times 340$

m) $\frac{4}{10} \times 30$ n) $\frac{6}{15} \times 30$ o) $2\frac{2}{3} \times 5$ p) $1\frac{1}{8} \times 40$ q) $2\frac{1}{3} \times 10$ r) $3\frac{7}{9} \times 30$

Equivalent Fractions

If you look at your answers to sums **a**, **m** and **n** in Exercise 2 you should find that they are all the same.
This suggests that the fractions $\frac{2}{5}$, $\frac{4}{10}$ and $\frac{6}{15}$ are all equivalent (the same).
This is in fact true and we can see why by looking at the diagrams below:



$\frac{2}{5}$ $\frac{4}{10}$ $\frac{6}{15}$

The amount shaded is the same in each diagram so they all show $\frac{2}{5}$.
But the extra line in the middle diagram shows that $\frac{4}{10}$ is the same as $\frac{2}{5}$.

Notice also that the 4 and the 10 in $\frac{4}{10}$ are twice the 2 and the 5 in $\frac{2}{5}$.
So we can get $\frac{4}{10}$ by multiplying the numerator and denominator of $\frac{2}{5}$ by 2: $\frac{2}{5} = \frac{4}{10}$.
Similarly we can get $\frac{6}{15}$ by multiplying numerator and denominator of $\frac{2}{5}$ by 3: $\frac{2}{5} = \frac{6}{15}$.

> In fact we can generate as many fractions as we like from a fraction by multiplying the top and bottom by any number we like.

If we multiply $\frac{2}{5}$ on the top and bottom by 2, 3, 4, 5, … etc. we get

$$\frac{4}{10}, \frac{6}{15}, \frac{8}{20}, \frac{10}{25}, \ldots \text{ etc.}$$

We call these equivalent fractions.

They also come under the Proportionately formula.

Example 9

Give 5 fractions equivalent to $\frac{3}{4}$.

Multiplying $\frac{3}{4}$ (top and bottom) by 2, 3, 4, 5 and 6 we get: $\frac{3}{4} = \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \frac{15}{20}, \frac{18}{24}$.

Example 10

Which of the fractions $\frac{8}{12}, \frac{10}{18}, \frac{16}{24}, \frac{14}{21}$ are equivalent to $\frac{2}{3}$?

$\frac{8}{12}$ is equivalent to $\frac{2}{3}$ because if we multiply 2 and 3 by 4 we get 8 and 12.
But $\frac{10}{18}$ is not equivalent to $\frac{2}{3}$ because we would need to multiply the top by 5 and the bottom by 6.
$\frac{16}{24}$ we can get from $\frac{2}{3}$ by multiplying top and bottom by 8, so it is equivalent, and $\frac{14}{21}$ is also equivalent since we can multiply the $\frac{2}{3}$ by 7.

Exercise 3

Give 4 equivalent fractions for each of the following:

a) $\frac{5}{3}$ b) $\frac{1}{4}$ c) $\frac{7}{9}$ d) $\frac{5}{6}$ e) $\frac{3}{2}$ f) $\frac{8}{10}$

g) Which of the following fractions are equivalent to $\frac{4}{5}$?
$\frac{12}{15}, \frac{8}{12}, \frac{6}{10}, \frac{20}{25}, \frac{16}{20}$

h) Two of the following fractions are equivalent: which are they and why?
$\frac{1}{5}, \frac{2}{12}, \frac{3}{12}, \frac{4}{20}, \frac{5}{10}$

i) Two of the following fractions are equivalent: which are they and why?
$\frac{5}{7}, \frac{6}{8}, \frac{7}{10}, \frac{8}{10}, \frac{9}{12}$

Simplifying Fractions

We can find as many fractions as we like equivalent to a given fraction, but there is always one that is simpler than all the others.

Given $\frac{2}{3}=\frac{4}{6}=\frac{6}{9}=\frac{8}{12}=\frac{10}{15}$ which are equivalent, $\frac{2}{3}$ is the simplest of them all.

> If we have to obtain the simplest fraction equal to a given fraction we look for the **highest common factor** of numerator and denominator and divide them by this number.

Example 11

Simplify $\frac{10}{15}$.
Since 10 and 15 can both be divided by 5, we divide them both by 5 and say: $\frac{10}{15}=\frac{2}{3}$

Example 12

Simplify $\frac{16}{12}$.
The HCF of 16 and 12 is 4, so we divide by 4.
We choose the largest divisor.

So $\frac{16}{12}=\frac{4}{3}$

Had we divided 16 and 12 by 2 instead we would get $\frac{16}{12}=\frac{8}{6}$ which is correct but does not give the simplest fraction. We would then need to observe that 6 and 8 can both be divided by 2, and then dividing we get $\frac{16}{12}=\frac{8}{6}=\frac{4}{3}$.
This gives the same answer, but it is a bit longer.

Simplifying fractions like this is also called canceling down, or putting into lowest terms.
It comes under the Vedic formula If the Samuccaya is the same, it is zero which simply means that any common factor can be cancelled out.

Exercise 4

Cancel down the following fractions (converting to mixed numbers where appropriate). Check your answer each time to see if it can be divided further:

a) $\frac{4}{10}$ b) $\frac{6}{9}$ c) $\frac{15}{20}$ d) $\frac{44}{55}$ e) $\frac{14}{21}$ f) $\frac{9}{6}$ g) $\frac{25}{10}$ h) $\frac{2}{18}$ i) $\frac{3}{18}$ j) $\frac{4}{18}$

k) $\frac{6}{18}$ l) $\frac{8}{18}$ m) $\frac{10}{18}$ n) $\frac{18}{18}$ o) $\frac{25}{30}$ p) $\frac{27}{30}$ q) $\frac{16}{24}$ r) $\frac{63}{54}$ s) $\frac{66}{39}$ t) $\frac{72}{39}$

Finding What Fraction One Number Is Of Another

If we compare the numbers 1 and 4 we see that 4 is 4 times 1, and that 1 is a quarter of 4.

If we are asked what fraction 1 is of 4, the answer is 1 is $\frac{1}{4}$ of 4.

Example 13

What fraction is 9 of 12?

The fraction is $\frac{9}{12}$ which cancels down to $\frac{3}{4}$.
So 9 is $\frac{3}{4}$ of 12

We simply put the first number on top of the fraction and the second on the bottom, and cancel if possible.

Example 14

What faction is x bananas of y bananas?

The answer is $\frac{x}{y}$.

Example 15

What fraction is 10 grams of 3kg?

Here we must be careful, we should not mix up the units.

Converting the 3kg into 3000 grams we write the fraction as $\frac{10}{3000}$ or $\frac{1}{300}$.

Exercise 5

Cancel down your answers to the following questions where appropriate:

a) What fraction is 5 of 20?

b) What fraction is 2 of 17?

c) What fraction is 20ml of 1liter?

d) What fraction is 45º of 180º?

e) What fraction is 1m 20cm of 6m?

f) What fraction is 10 minutes of 1 ½ hours?

g) What fraction is 600m of 3km?

h) What fraction of 1 minute is 15 seconds?

i) What fraction of 3 feet is 4 inches?

j) What fraction is 2 zodiac signs of the Rasi chakra?

Exercise 6

Ask your teacher for more exercises.

32 Planets

Our solar system consists of 9 planets circling the Earth and various other objects.

Some planets are small compared with other much larger planets.

In this chapter you will get an idea of the relative sizes by drawing a circle to represent each planet.

Planet sizes according to Surya Siddhanta

Planet	Orbit (in *yojanas*)	Diameter (in *yojanas*)
Moon	324,000	480.00
Sun	4,331,500	6,500.00
Mercury	4,331,500	601.60
Venus	4,331,500	802.13
Earth	0	1,600.00
Mars	8,146,909	754.34
Jupiter	51,375,764	8,324.80
Saturn	127,668,255	14,776.00

* The data are from Surya Siddhanta and the measurement for 1 *yojana* is 5 miles.

Exercise 1

Ask your teacher for exercises.

33 Simple Equations (*yavattavat*)

You are familiar with the solution of equations like 2x – 5 = 7 and $\frac{3x}{5} + 4 = 1$.
In this chapter we extend the range of equations we can handle to those with fractional answers and other types of equation. We begin with some revision.

Exercise 1

Solve the following mentally:

a) 2x – 5 = 7 b) 3x + 4 = 25 c)2x + 13 = 5 d) 5x – 1 = $\overline{16}$

e) $\frac{x}{3} - 3 = 4$ f) $\frac{2x}{3} + 4 = 10$ g) $\frac{x}{2} - 3 = 7$ h) $\frac{3x}{4} + 4 = 1$

Some Variations

Equations may not always be in the standard forms above.

Example 1

Solve 23 = 2 + 3x.

This is just the same as 3x + 2 = 23 and so we mentally take the 2 from the 23 and divide by 3.
We get x = 7.

Example 2

Solve 2x + 13 + x = 4.

Here we mentally simplify the left side of the equation by combining the 2x and the x to mentally get 3x + 13 = 4, so that x = $\overline{3}$.

Example 3

Solve 16 = 5 – 2x.

Here we see -2x on the right side of the equation.
We mentally put this on the left side because the minus then becomes a plus,
and take the 16 to the right to get 2x = 5 – 16.

● Make sure you understand what has happened here: the -2x is transposed to 2x on the other side, and the 16 (which is +16) is transposed to -16 on the other side. The 5 is unaltered.
This gives x = -5 ½.

If you like you can take the 5 in 16 = 5 – 2x to the left to get 16 – 5 = -2x which becomes 11 = -2x. Then dividing 11 by -2 we get x = -5 ½ as before.

Exercise 2

Solve the following:

a) 3 + 4x = 23	b) 17 = 1 + 2x	c) 2x + 3x = 25	d) x + 1 + x = 11
e) x – 3 + 3x = 13	f) 30 = 2x + 3x	g) 7 + x = 50 – 3	h) 3x + 2 + 2x – x = 30
i) 5 = 1 – x	j) 18 = 22 – 2x	k) 16 = 1 – 5x	l) 7 – 3x = 19

Fractional Answers

Example 4

Solve 3x – 5 = 8.

Using *Transpose and Apply* we first get 3x = 13,
then since 3 does not divide exactly into 13 we write $x = \frac{13}{3}$

$x = 4\frac{1}{3}$ is also correct but we will leave answers in top-heavy form for the rest of this chapter.

Example 5

Solve 18x = 2.

The answer is simply $x = \frac{2}{18}$ which cancels down to $x = \frac{1}{9}$.

Example 6

Solve $\frac{x}{3} = \frac{4}{5}$.
Since we need x here we transpose the 3: $x = 3 \text{ x } \frac{4}{5}$.
So $x = \frac{12}{5}$.

It is sometimes useful to cross-multiply in equations like the one in Example 6.
This means we multiply diagonally and equate the two products: $x \times 5 = 4 \times 3$,
which gives 5x = 12 and $x = \frac{12}{5}$ again.
This is particularly useful with equations like the one below.

Example 7

Solve $\frac{5}{x} = \frac{2}{3}$
Cross-multiplying gives 2x = 15 and $x = \frac{15}{2}$.

Exercise 3

Solve the following (cancel but leave as top-heavy fractions where appropriate):

a) $3x = 5$ b) $8x = 1$ c) $3x - 4 = 7$ d) $5x + 4 = 5$ e) $3 + 2x = 10$

f) $9 = 1 + 5x$ g) $3x + 12 = 4$ h) $2 - 3x = 6$ i) $4x + 3 = \bar{6}$ j) $2 - 4x = \bar{5}$

k) $\frac{3x}{2} + 1 = 5$ l) $\frac{4x}{5} - 2 = 4$ m) $\frac{x}{5} = \frac{8}{3}$ n) $\frac{x}{6} = \frac{3}{4}$ o) $\frac{2x}{3} = \frac{1}{4}$

p) $\frac{2}{1} = \frac{3x}{5}$ q) $\frac{3}{x} = \frac{2}{7}$ r) $\frac{2}{3x} = \frac{6}{5}$ s) $\frac{1}{3} = \frac{4}{3x}$ t) $\frac{1}{5} = \frac{1}{3x}$

Two x Terms

So far none of our equations have had an x term on both sides of the equation.

Example 8

Solve $5x + 3 = 3x + 17$.

The method here is to collect all the x terms on one side of the equation and all the other terms on the other side.

In the above equation we take the 3x to the left side and the 3 to the right.

This gives $5x - 3x = 17 - 3$.
So $2x = 14$,
and $\underline{x = 7}$.

Note carefully how the terms which have changed side have also changed sign. The other terms are left unchanged.

Exercise 4

Solve the following:

a) $7x + 1 = 5x + 11$ b) $8x + 3 = 3x + 23$ c) $4x + 7 = x + 16$

d) $5x + 1 = 2x + 19$ e) $2x + 3 = x + 1$ f) $10x + 17 = 5x + 2$

These equations can also be solved mentally. We can see how many x's there will be on the left and what the number on the right will be when we have transposed. Then we just divide the number on the right by the number on the left.

So for Example in question (a) above we see there will be 2x on the left when the 5x is taken over, and that there will be 10 on the right when the 1 is taken over.
Then we just divide 10 by 2 to get x=5.

In the next exercise write down only the answer.

Exercise 5

Solve the following mentally:

a) $5x + 3 = 3x + 15$ b) $3x + 1 = x + 21$ c) $7x + 5 = 4x + 20$

d) $8x + 3 = 3x + 28$ e) $10x + 11 = 3x + 32$ f) $9x + 17 = x + 81$

Example 9

Solve $7 - 2x = x - 5$.

Here seeing the -2x it is best to collect the x terms on the right.
Then the -2x will become +2x on the right, and the -5 will be +5 on the left.
This gives $12 = 3x$ (mentally).

So <u>$x = 4$</u>.

Example 10

Solve $2(3x + 4) = 2x + 20$.

Mentally we see there will be 4x on the left and 12 on the right.
So <u>$x = 3$</u>.

Example 11

$6x + 5 = 4x - 21$.

There will be 2x on the left and -26 on the right (since $-21-5 = -26$).
So <u>$x = -13$</u>.

Exercise 6

Solve the following mentally:

a) $7x - 5 = 4x + 10$ b) $5 + 4x = 13 + 2x$ c) $7x + 3 = 15 + x$

d) $5x - 21 = x - 1$ e) $6x + 1 = 4x - 3$ f) $5x - 1 = 3x + 9$

g) $8x - 9 = 5x + 12$ h) $8x - 2 = 6x - 23$ i) $10x + 1 = 25 - 2x$

j) $2(3x + 1) = x + 27$ k) $3(x - 3) = 2x + 8$ 1) $14 - 3x = x + 10$

m) $17 - 5x = 8 - 2x$ n) $9x - 3 = -x + 57$ o) $4x - 7 = x + 9$

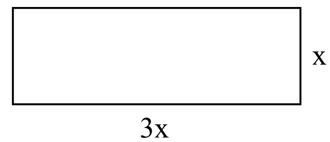
<u>Forming Equations</u>

Equations are very useful in solving problems when the problem can be put into mathematical form.

Example 12

A rectangle is 3 times as long as it is wide. If its perimeter is 52cm find its length and width.

The fact that the rectangle is 3 times as long as it is wide can be put into mathematical form by calling the shorter side **x**, because we can then call the longer side **3x**.



Since the perimeter is given as 52 we can write the equation:

$$3x + x + 3x + x = 52.$$

Which means 8x = 52 and x = 6½

So the answer to the question is <u>19½ and 6½</u>.

Putting a problem into mathematical form like this is often very useful and the first step is to decide what can usefully be called x. We then interpret the rest of the given information in terms of x, as shown above.

Example 13

Three consecutive numbers add up to 114. What are they?

If we call the first number **n**, then **n+1** must be the next number (since they are consecutive) and **n+2** is the third. It is usual to use n rather than x if it is known to be a whole number.

We are told that they add up to **114** so: $n + n + 1 + n + 2 = 114$.
Therefore 3n + 3 = 114 and n = 37.

So the numbers are <u>37, 38 and 39</u>.

Example 14

Find two consecutive odd numbers such that the sum of 5 times the smaller and twice the greater is 95.

We have seen that 2n +1 represents an odd number so the next odd number will be **2n + 3**.

So the question says that $5(2n + 1) + 2(2n + 3) = 95$.
Multiplying out the brackets gives 10n + 5 + 4n + 6 = 95.
So 14n + 11 = 95 and n = 6.

Putting this value into 2n + 1 and 2n + 3 gives <u>13 and 15</u>.

Exercise 7

Express the following as an equation and find the number(s):

a) When 30 is added to 3 times a number the result is 66.

b) A rectangle is twice as long as it is wide and its perimeter is 66.

c) 4 consecutive numbers add up to 142.

d) Three consecutive even numbers add up to 72 (call the first even number 2n).

e) 2 consecutive odd numbers add up to 104.

f) There are two consecutive numbers and the greater added to 3 times the smaller makes 53.

g) A room is 3m longer than its width and its perimeter is 38m.

Quadratic Equations

Example 15

Solve the equation $x^2 = 25$.

This says that a number squared comes to 25.
The number is clearly 5 as $5^2 = 25$. So $x = 5$.

However this is not the whole answer because $(-5)^2 = 25$ as well.
So the full answer is $\underline{x = 5 \text{ or } x = -5}$.

This can also be written more briefly as $\underline{x = \pm 5}$ which means exactly the same.

Example 16

Solve $3x^2 = 48$.

We divide both sides by 3 to get $x^2 = 16$.

$\underline{\text{So } x = \pm 4}$.

Example 17

Solve $x^2 + 3 = 15$.

Here we take 3 from both sides to get $x^2 = 12$.
But in this case 12 is not a square number so we write $x = \pm\sqrt{12}$

Example 18

Solve $2x^2 - 100 = 1700$.

Here we have to add 100, divide by 2 and take the square root.
This can all be done mentally to get $\underline{x = \pm 30}$.

Equations involving x^2, like this, are called **quadratic equations**.

Exercise 8

Solve the following:

a) $x^2 = 1600$ b) $3x^2 = 300$ c) $x^2 + 5 = 69$ d) $x^2 - 600 = 3{,}000$

e) $5x^2 = 45$ f) $4x^2 - 4 = 200$ g) $9x^2 - 9 = 0$ h) $2x^2 + 3 = 53$

i) $4x^2 - 7 = 2$ j) $x^2 - 3 = 1222$ k) $2x^2 = 882$ l) $\frac{x^2}{3} = 27$

Exercise 8
Ask your teacher for more exercises.