Vedic Mathematic

Laghu

Bhaktivedanta Academy

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1 Nikhilam Multiplication

The Sutra Nikhilam Navatascharaman Dasatah means *All from 9 and the Last from* we have met this Sutra before.

Usually a sum like 88 × 98
is considered especially difficult because of the large figures, 8 and 9.

But since the numbers 88 and 98 are close to the base of 100 we may think that there ought to be an easy way to find such a product.

EXAMPLE 1

88 − 12	88 is 12 below 100, so we put -12 next to it;
98 − 2	98 is 2 below 100, so we put -2 next to it;
86 / 24	We call the 12 and 2 **deficiencies** as the numbers 88 and 98 are deficient from the unity of 100 by 12 and 2.

The answer 8624 is in two parts: 86 and 24.
The 86 is found by taking one of the deficiencies from the other number:
That is 88 – 2 = 86 or 98 – 12 = 86 (whichever you like),
And the 24 is simply the product of the deficiencies: 12 × 2 = 24.
So 88 × 98 = 86. It could hardly be easier.

EXAMPLE 2

For 93 × 96 we get deficiencies of 7 and 4, so

93 − 07
96 − 04
89 / 28

The differences from 100 are 7 and 4,
93 – 4 = 89 or 96 – 7 = 89,
and 7 × 4 = 28.

In fact once we have got the deficiencies we apply the *Vertically and Cross-wise sutra*:
we **cross-subtract** to get the left-hand part of the answer and

EXAMPLE 3

For 98 × 97:

$$\begin{array}{l} 98\ -\ 02 \\ \underline{97\ -\ 03} \\ \underline{95\ /\ 06} \end{array}$$

Note the zero inserted here: the numbers being multiplied are near to 100, so two digits are required on the right as in the other examples.

EXAMPLE 4

For 89 × 89:

$$\begin{array}{l} 89\ -\ 11 \\ \underline{89\ -\ 11} \\ \underline{78\ /_{1}\ 21} = \underline{7921.} \end{array}$$

Here the numbers are each 11 below 100, and 11 × 11 = 121, a 3-figure number. The hundreds digit of this is therefore carried over to the left.

EXERCISE 1

Multiply the following:

a) 94 × 94 b) 97 × 89 c) 87 × 99 d) 87 × 98 e) 87 × 95 f) 95 × 95
g) 79 × 96 h) 98 × 96 i) 92 × 99 j) 88 × 88 k) 97 × 56 l) 97 × 63
m) find a way to get 92 × 196?

The most efficient way to do these sums is to take one number and subtract the other deficiency from it. Then multiply the deficiencies together, mentally adjusting the first part of the answer if there is a carry figure.
This is so easy it is really just mental arithmetic.

EXERCISE 2

Multiply these numbers mentally, just write down the answer:

a) 87	b) 79	c) 98	d) 94	e) 96	f) 88	g) 89
97	98	93	95	96	96	98
—	—	—	—	—	—	—
—	—	—	—	—	—	—

h) 93	i) 93	j) 97	k) 96	l) 95	m) 89	Find the missing number
96	99	97	67	75	??	
					8544	

Multiply these numbers mentally, just write down the answer:

a 87	**b** 79	**c** 98	**d** 94	**e** 96	**f** 88	**g** 89
97	98	93	95	96	96	98

h 93	**i** 93	**j** 97	**k** 96	**l** 95	**m** 89	find the
96	99	97	67	75	??	missing
					8544	numbers

OTHER BASES

EXAMPLE 5

Find 568 × 998.

In this sum the numbers are close to 1000, and the deficiencies are 432 and 2.
The deficiency for 568 is found by applying the Sutra: *All from 9 and the Last from 10*

568 - 432
998 - 2 The method here is just the same, but we allow 3 figures
566 / 864 on the right as the base is now 1000.

The number of spaces needed on the right is the number of 0's in the base number.

The differences of the numbers from 1000 are 432 and 2.
Then cross-subtracting: 568 - 2 = 566,
And vertically: 432 × 2 = 864.

EXAMPLE 6

Find 58776 × 99998.

Even large numbers like this are easily and mentally multiplied by the same method.

58776 - 41224
99998 - 2
58774 / 82448

EXAMPLE 7

Find 7 × 8.

In the Vedic system tables above 5×5 are not really essential:

Exactly the same method gives us 7 × 8 = 56.

$$\begin{array}{l} 7 - 3 \\ \underline{8 - 2} \\ \underline{5 / 6} \end{array}$$

EXERCISE 3

Multiply the following mentally:

a 667 × 998
b 768 × 997
c 989 × 998
d 885 × 997
e 883 × 998
f 467 × 998
g 891 × 989
h 8888 × 9996
i 6999 × 9997
j 90909 × 99994
k 78989 × 99997
l 9876 × 9989

NUMBERS ABOVE THE BASE

Suppose now that the numbers are not both below a base number as in all the previous examples, but above the base.

EXAMPLE 8

103 × 104 = <u>10712</u>

$$\begin{array}{l} 103 + 3 \\ \underline{104 + 4} \\ \underline{107 / 12} \end{array}$$

This is even easier than the previous examples, but the method is just the same.
The differences from the base are +3 and +4 because the numbers are now **above** the base.

103 + 4 = 107 or 104 + 3 = 107, and 4 × 3 = 12.

So now we **cross-add,** and multiply vertically.

EXAMPLE 9

12 × 13 = <u>156</u> (12+3=15, 2×3=6)

EXAMPLE 10

$1234 \times 1003 = \underline{1237702}$ (1234+3=1237, 234×3=702)

EXAMPLE 11

$10021 \times 10002 = \underline{100230042}$ (10021+2=10023, 0021×2=0042)

With a base of 10000 here we need 4 figures on the right.

- Check that you agree with all the examples above and then do the following exercise mentally.

EXERCISE 4

a 133 × 103
b 107 × 108
c 171 × 101
d 102 × 104
e 123 × 102
f 14 × 12
g 18 × 13
h 1222 × 1003
i 1051 × 1007
j 15111 × 10003
k 125 × 105
l 10034 × 10036

PROPORTIONATELY

The *Proportionately* formula considerably extends the range of this multiplication method.

EXAMPLE 12

$213 \times 203 = \underline{43239}$

$$\begin{array}{rr} & 213 + 13 \\ & \underline{203 + \ \ 3} \\ 2 \times & \underline{216\ /39} = \underline{43239} \end{array}$$

We observe here that the numbers are not near any of the bases used before: 10, 100, 1000 etc.. But they are close to 200, with differences of 13 and 3 as shown above.

The usual procedure gives us 216/39 (213+3=216, 13×3=39).

Now since our base is 200 which is 100×2 we multiply **only the left-hand part** of the answer by 2 to get 43239.

EXAMPLE 13

29 × 28 = <u>812</u>

The base is 30 (3×10),
and the deficiencies are -1 and -2.
Cross-subtracting gives 27,
then multiplying vertically on the right we get 2,
and finally 3×27 = 81.

$$\begin{array}{rl} & 29 - 1 \\ & \underline{28 - 2} \\ 3 \times & \underline{27\ /\ 2} = 812 \end{array}$$

So these are just like the previous sums but with an extra multiplication (of the left-hand side only) at the end.

EXAMPLE 14

Find 33 × 34.

In this example there is a carry figure:

$$\begin{array}{rl} & 33 + 3 \\ & \underline{34 + 4} \\ \underline{3 \times} & \underline{37\ /\ _{1}2} = 111\ /_{1}2 = \underline{1122} \end{array}$$

Note that since the right-hand side does not get multiplied by 3 we multiply the left-hand side by 3 before carrying the 1 over to the left.

EXAMPLE 15

88 × 49 = $\frac{1}{2}$(88x98) = $\frac{1}{2}$(8624) = 4312.

This example shows a different application of *Proportionately.*
In 88 × 49 the numbers are not both close to 100,
but since twice 49 is 98 we can find 88 × 98 and halve the answer at the end.

EXERCISE 5

Multiply mentally:

a 41 × 42 **b** 204 × 207 **c** 321 × 303 **d** 203 × 208 **e** 902 × 909

f 48 × 47 **g** 188 × 196 **h** 199 × 198 **i** 189 × 194 **j** 207 × 211

k 312 × 307 **l** 5003 × 5108 **m** 63 × 61 **n** 23 × 24 **o** 79 × 77

p 44 x 98 q 48 x 97 r 192 x 97

SQUARING NUMBERS NEAR A BASE

This is especially easy and is for squaring numbers which are near a base.
You will recall that squaring means that a number is multiplied by itself (like 96 x 96).
This method is described by the sub-formula *Reduce (or increase) by the Deficiency and also set up the square.*

EXAMPLE 16

$96^2 = 92/16$

96 is 4 below 100, so we reduce 96 by 4, which gives us the first part of the answer, 92.
The last part is just $4^2 = 16$, as the formula says.

EXAMPLE 17

$1006^2 = 1012/036$

Here 1006 is increased by 6 to 1012, and $6^2 = 36$: but with a base of 1000 we need 3 figures on the right, so we put 036.

EXAMPLE 18

$304^2 = 3 \times 308/16 = 92416$

This is similar but because our base is 300 the left-hand part of the answer is multiplied by 3.

1 In this last example we can write $304^2 = 9/24/16$ in which the answer has been split into three parts. Can you see how to get these parts from the figures in 304?
Do you think this will always work with squaring numbers like a0b?

EXERCISE 6

Square the following: **a** 94 **b** 103 **c** 108 **d** 1012 **e** 98 **f** 88 **g** 91
h 10006 **i** 988 **j** 997 **k** 9999 **l** 9989 **m** 111 **n** 13 **o** 212 **p** 206
q 302 **r** 601 **s** 21 **t** 72 **u** 4012 **v** 511 **w** 987

MULTIPLYING NUMBERS NEAR DIFFERENT BASES

EXAMPLE 19

9998 × 94 = 9398/12

Here the numbers are close to different bases: 10,000 and 100,
and the deficiencies are -2 and -6.
We write, or imagine, the sum set out as shown:

9998 - 2
94 - 6
9398 / 12

- Can you see how the two parts of the answer are found?

It is important to line the numbers up as shown because the 6 is not subtracted from the 8, but from the 9 above the 4 in 94. That is, the second column from the left here.

So **9998 becomes 9398.**

Then multiply the deficiencies together: 2×6 = **12**.

Note that the number of figures in the right-hand part of the answer corresponds to the base of the lower number (94 is near 100, therefore there are 2 figures on the right).

You can see why this method works by looking at the sum 9998 x 9400, which is 100 times the sum done above:

9998 - 0002
9400 - 600
9398 / 1200

- Check that you agree with this calculation.

Now we can see that since 9998 X 9400 = 93981200,
then 9998 x 94 = 939812.

This also shows why the 6 is subtracted in the second column from the left.

EXAMPLE 20

10007 × 1003 = 10037021

Lining the numbers up:

10007 + 007
1003 + 003
10037 / 021

we see that we need three figures on the right and that the surplus, 3, is added in the 4th column, giving 10037.

EXERCISE 7

Find:

a 97 × 993
b 92 × 989
c 9988 × 98
d 9996 × 988
e 103 × 1015
f 106 × 1012
g 10034 × 102
h 1122 × 104

A SUMMARY

Here we can summarise the various methods of multiplication and squaring encountered so far.

1. Multiplying by 4, 8 etc. we can just double twice, 3 times etc.. E.g. 37×4.
2. We can use doubling to extend the multiplication tables. E.g. 14×8.
3. We can multiply from left to right using *On the Flag*. E.g. 456×3.
4. We can square numbers ending in 5, using *By One More than the One Before*. E.g. 35^2.
5. With the same Sutra we can multiply numbers that have the same first digit and whose las figures add up to 10. E.g. 72×78.
6. We can use *All from 9 and the Last from 10* for multiplying numbers near a base.

   E.g. 98×88, 103×104, 203×204.
7. The same Sutra can be used for squaring numbers near a base. E.g. 97^2, 1006^2, 203^2.
8. And we can also multiply numbers near different bases. E.g. 998×97.

EXERCISE 8

The following exercise contains a mixture of all these different types of multiplication:

a 654 × 3
b 86 × 98
c 97 × 92
d 73 × 4
e 7 × 22
f 16 × 24
g 798 × 997
h 8899 × 9993
i 106^2
j 996^2
k 103 × 109
l 123 × 104
m 203 × 209
n 188 × 197
o 85^2
p 73 × 77
q 32 × 33
r 2004 × 2017
s 9997 × 98
t 1023 × 102
u 84 × 86
v 28 × 54
w 303 × 307
x 93^2
y 1011^2
z 403^2

2 Spirals

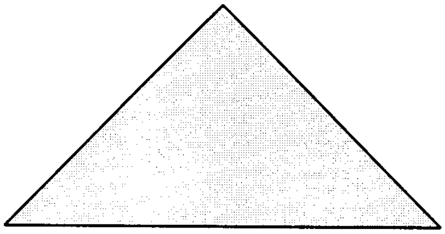
The spiral is a very beautiful and important shape. It occurs frequently in nature, in shells, flowers and the structure of plants.

There are different kinds of spirals. The one found in nature is usually the equiangular spiral. The principle behind the growth of an equiangular spiral is that the shape is constant- any part of the curve is an exact enlargement of an earlier part. As the creature that lives in a shell enlarges its home it keeps it the same shape but bigger.

THE ISOSCELES RIGHT-ANGLED TRIANGLE

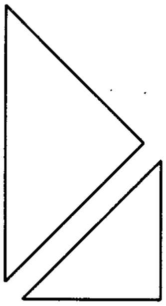
There is only one shape of triangle which is both isosceles and right-angled.

● What will the angles be in a triangle which is both isosceles and right-angled?



Suppose the top angle here is 90° then the other two angles must be the equal ones.
And since we can see the triangle as one half of a square the two base angles must be 45° each.
Though the shape of this triangle is fixed its size and its position may vary.

For example both of these triangles are isosceles right-angled triangles The longest side of aright-angled triangle is called the hypotenuse.



Note the way the larger triangle almost stands on the hypotenuse of the smaller triangle.

EXERCISE 1

a) Take a sheet of graph paper and with the longer edge at the bottom draw a horizontal line 1 cm long on one of the bold lines about 8cm from the left edge and about 8cm from the top edge.

b) This line is the base of an isosceles right-angled triangle with the right angle at the right end of the line. Complete the triangle.

c) Another isosceles right-angled triangle must now be drawn with its base on the hypotenuse of the first triangle (rather like the diagram above, but the lines will touch).
Be careful and use the lines on your page to guide you.

d) The second triangle is an enlarged version of the first one and is joined onto it.
We are going to repeat this procedure: add a third triangle to the hypotenuse of the second one. It must be the same shape as the others, with the right angle on the right of the base.
You may prefer to turn your sheet so that the right-hand edge is at the bottom.

e) Continue this until you have 8 triangles altogether. You should then have a kind of spiral shape.

f) The next triangle will go off your page so we cannot draw any bigger triangles.
It is possible however to draw some smaller ones. See if you can draw a triangle below the first one so that its hypotenuse is the base of the first triangle.
If you can do this draw in some more even smaller triangles.

g) Note how the base of each triangle is rotated 45° to become the larger base of the next triangle.
Put a dot at the top corner of each triangle. Draw a smooth spiral through all these dots and when it looks right go over it with a coloured pencil.

This shows the kind of equiangular growth discussed at the beginning of this chapter.

SPIRALS FROM SQUARES

EXERCISE 2

a) On a new graph page draw a square with sides 2cm near the centre of the sheet. Use the bold lines on the graph paper. We will call this Square 3. In your book write down the area of Square 3.

b) Put a dot in the middle of each side of your square and join them up to make a diamond shape. Is this shape also a square?

c) The sides of your smaller shape are equal and so are the four angles, so it must be a square. Call this Square 2. What is the area of this square?

d) Looking carefully at the four squares inside the larger square you should see that the diagonals you drew inside it cut each of these four squares in half. It follows that the inner square has half the area of the outer one, that is 2cmz. Draw another square inside Square 2 by joining the middle of the sides of Square 2. By the same logic as above the area of this square must also be half of the area of Square 2. So its area is 1 cmz. Check with your ruler that this is correct.

e) Now we have 3 squares inside each other. Starting with the inner square (Square 1) it is as though Square 1 is rotating and growing to become the next square. Draw a fourth square outside Square 3 so that the corners of Square 3 are in the middle of each side.

f) Add another square outside Square 4 and continue doing this as far as you can. You will probably have 9 squares.

g) Complete a list of the areas of each square in your book.

h) Next we can draw in a spiral. Put a dot in the top left-hand corner of Square 1. The first spiral will go clockwise from here. Imagine Square 1 turning clockwise and growing so that it becomes Square 2: the dot will now be at the top of Square 2. Put a dot there. Next Square turns onto Square 3 and the dot is now at the top right of Square 3. Continue to add dots until you arrive at a dot in the top left corner of Square 9. Join the dots (first with a pencil and then in colour) with a smooth spiral curve.

i) Now you have a choice: you can either draw in three more clockwise spirals starting at each of the other three corners of Square 1, or you can start again at the top left-hand corner of Square 1 but turn anti-clockwise instead of clockwise.

AN INFINITE SUM

EXERCISE 3

a) On a fresh graph page draw a square 16cm by 16cm.
The method we are going to use here is to halve a shape (a square or rectangle), put a dot in the centre of one part and halve the other part, and keep repeating this.

b) Halve your square by drawing a vertical line down the middle and put a dot in the exact centre of the left-hand half.

c) Halve the right-hand rectangle (by drawing a horizontal line) and put a dot in the centre of the upper half.

d) Halve the lower square (by drawing a vertical line) and put a dot in the centre of the right-hand half.

e) Halve the left-hand rectangle, and put a dot in the centre of the lower half.
Continue this process- you can go back to b above if you like and keep repeating steps b,c,d,e until you have at least 9 dots.

f) Next you can carefully draw in the spiral that goes through all the dots in order.

g Now suppose that the side of the large square is called 1 unit. This means that the area of the large square is 1 square unit. Each dot you have drawn is in the centre of a square or rectangle. Starting with the large rectangle on the left write down the area of each shape which has a dot in its centre. The first rectangle, for example must have an area of $\frac{1}{2}$ a square unit.

h Each area will be half of the previous area and we can see that since the spiral goes on forever there are an infinite number of areas.
The area of the big square is the total of the areas of all the shapes inside it.
This is: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \ldots\ldots$ where the dots mean that the fractions go on forever.
But since the area of the big square is 1 we can write:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \ldots\ldots = 1,$$

and we have found the total of an infinite number of fractions!

3 Fractions and Decimals

In this chapter we are going to study how decimals and fractions are related, and how to convert from one to the other.

CONVERTING TO FRACTIONS

EXAMPLE 1

A decimal like 0.6 is easily converted into a fraction: we know that $0.6 = \frac{6}{10}$.

And this can be cancelled down: $\frac{6}{10} = \frac{3}{5}$.

So the simplest fraction equal to 0.6 is $\frac{3}{5}$.

EXAMPLE 2

Convert 0.05 to a fraction.

Since the 5 is in the hundredths position: $0.05 = \frac{5}{100} = \frac{1}{20}$.

EXAMPLE 3

Convert 0.15 to a fraction.

The 1 here is in the tenths position and the 5 is in the hundredths position.
But we can think of the 1 tenth as being 10 hundredths- rather like this:

$$0.15 = 0.0_{1}5$$

then we can see that $0.15 = \frac{15}{100} = \frac{3}{20}$.

EXAMPLE 4

Convert 23.4 to a fraction.

Here we think of the whole part and the fraction separately:
we have 23 whole ones and $0.4 = \frac{4}{10} = \frac{2}{5}$. So $\underline{23.4 = 23\frac{2}{5}}$.

EXERCISE 1

Convert the following into fractions (cancel down as much as possible):

a 0.3 **b** 0.4 **c** 0.04 **d** 0.07 **e** 0.005 **f** 0.0001

g 0.25 **h** 0.35 **i** 0.16 **j** 0.17 **k** 0.055 **l** 0.032

m 0.0011 **n** 1.3 **o** 2.5 **p** 3.7 **q** 8.01 **r** 2.2

s 20.1 **t** 10.01 **u** 0.123 **v** 0.222 **w** 0.444

CHANGING FRACTIONS TO DECIMALS

EXAMPLE 5

Convert $\frac{7}{100}$ and $\frac{17}{100}$ and $\frac{17}{50}$ into decimal form.

We know that $\frac{7}{100} = 0.07$ and $\frac{17}{100} = 0.17$ and $\frac{17}{50} = \frac{34}{100} = 0.34$.

This method works well for fractions which have base numbers (10, 100, 1000 etc.) in the denominator, or which can easily be multiplied up as in the 3rd case above.

EXAMPLE 6

Convert $\frac{3}{4}$ into a decimal.

We know that a fraction is unchanged by multiplying or dividing the top and bottom by the same number. For example $\frac{2}{3} = \frac{4}{6}$ and $\frac{20}{25} = \frac{4}{5}$.

If we divide the top and bottom of $\frac{3}{4}$ by 4 (the denominator) we get $\frac{3 \div 4}{4 \div 4}$ which is $\frac{3 \div 4}{1}$ which is just $3 \div 4$.

In other words we can convert a fraction into a decimal by dividing the numerator by the denominator.

So **we write 3 as 3.00** and divide:

Dividing 3 by 4 gives 0 remainder 3: $\begin{array}{l} 4)\underline{3.{}_300} \\ \quad 0. \end{array}$

Next we divide 4 into 30 giving 7 remainder 2: $\begin{array}{l} 4)\underline{3.{}_30{}_20} \\ \quad 0.\ 7 \end{array}$

Finally 4 into 20 goes 5 times: $\begin{array}{l} 4)\underline{3.{}_30{}_20} \\ \quad 0.\ 7\ 5 \end{array}$

The division stops here and so we have $\frac{3}{4} = \underline{0.75}$

Alternatively, we can multiply the top and bottom of $\frac{3}{4}$ by 25 to get $\frac{3\times25}{4\times25} = \frac{3}{4} = \underline{0.75}$.

EXAMPLE 7

Convert $\frac{7}{5}$ into a decimal.

We set up the sum: $5)\underline{7.00}$ and continue just as in ordinary division until there is nothing left to divide.

It is not important how many zeros we put after the 7, we can always add more as we go along if we need to.

The division gives $\begin{array}{l} 5)\underline{7.{}_200} \\ \quad 1.\ 4 \end{array}$ so $\frac{7}{5} = \underline{1.4}$.

An alternative method here would be to change $\frac{7}{5}$ to $\frac{14}{10}$ which equals 1.4.

EXERCISE 2

Convert to decimals without using division:

a $\frac{9}{10}$ **b** $\frac{43}{100}$ **c** $\frac{3}{50}$ **d** $\frac{36}{1000}$ **e** $\frac{3}{20}$ **f** $\frac{13}{5}$ **g** $\frac{1}{25}$

Convert to decimals using division:

h $\frac{3}{5}$ **i** $\frac{5}{4}$ **j** $\frac{3}{8}$ **k** $\frac{13}{8}$ **l** $\frac{31}{4}$ **m** $\frac{125}{8}$

COMPARING FRACTIONS AND DECIMALS

EXAMPLE 8

Which is greater: 0.4 or $\frac{3}{8}$?

To compare these we convert $\frac{3}{8}$ to a decimal.

We get $\frac{3}{8} = 0.375$ by division and we can now see that this is less than 0.4 (if this not clear think of 0.4 as 0.400, then 375 is clearly less than 400).

EXERCISE 3

Decide in each case which is greater:

a $\frac{21}{50}$ or 0.4
b $\frac{13}{20}$ or 0.62
c $\frac{7}{4}$ or 1.8
d $\frac{23}{50}$ or $\frac{9}{20}$
e $\frac{3}{5}$ or 0.71
f $\frac{9}{8}$ or 1.2
g $\frac{4}{5}$ or 0.75
h $\frac{7}{2}$ or $\frac{17}{5}$
i Arrange the following in ascending order: 0.2 $\frac{1}{4}$ 0.65 $\frac{3}{5}$.

All the fractions dealt with above give decimals which divide exactly.
They are called Terminating Decimals.

RECURRING DECIMALS

EXAMPLE 9

Convert $\frac{1}{3}$ to a decimal.

On division we find:

$$3\underline{)1.{}_{1}0{}_{1}0{}_{1}0}$$
$$0.\ 3\ \ 3\ \ 3\ \ldots$$

We find here that the decimal goes on, giving an endless series of 3's.

Although the 3's go on forever it is possible to express this by writing $\frac{1}{3} = 0.\dot{3}$

The dot on top of the 3 indicates that it repeats itself endlessly.
This type of decimal is called a **Recurring Decimal.**

EXAMPLE 10

Convert $\frac{7}{6}$ to a decimal.

Division gives: $6\overline{)7.{}_{1}0{}_{4}0{}_{4}0}$

$$1.\ 1\ 6\ 6\ldots$$

Here we find that a 6 repeats itself indefinitely after the first two figures of the answer.
So we write $\frac{7}{6} = 1.1\dot{6}$.

Similarly $\frac{8}{300}$ can be found by first dividing numerator and denominator by 100 to give $\frac{0.08}{3}$.

EXERCISE 4

Convert to recurring decimals:

a $\frac{2}{3}$ **b** $\frac{4}{9}$ **c** $\frac{1}{6}$ **d** $\frac{13}{9}$ **e** $\frac{412}{3}$ **f** $\frac{4}{15}$ **g** $\frac{7}{90}$ **h** $\frac{13}{300}$

> Whether or not a decimal recurs or terminates depends only on the denominator:
> e.g. since $\frac{1}{4}$ terminates, so will $\frac{3}{4}$, $\frac{7}{4}$.

Similarly $\frac{1}{3}$ gives a recurring decimal so $\frac{2}{3}$, $\frac{4}{3}$, $\frac{5}{3}$ etc. will also be the recurring type, but $\frac{3}{3}$, $\frac{6}{3}$ etc. will not recur because $\frac{3}{3} = 1$, $\frac{6}{3} = 2$ and so on, so given a fraction it is best to first check it is in its lowest terms.

BLOCK RECURRERS

Recurring decimals may be of the type that recur like those above, or they may be of another type:

EXAMPLE 11

Convert $\frac{2}{7}$ into a decimal.

We divide as usual: $7\overline{)2.{}_{2}0{}_{6}0{}_{4}0{}_{5}0{}_{1}0{}_{3}0{}_{2}0{}_{6}0}$

$$0.\ 2\ 8\ 5\ 7\ 1\ 4\ 2.\ldots$$

We find the numbers 285714 appearing, then a remainder of 2. Since 20÷7 is how we started the division we find the same 2 which started the decimal coming up again. ☞

It follows that the same six figures will repeat themselves: we have a block of figures repeating themselves.

We write the answer to this as: $\frac{2}{7} = 0.\dot{2}\,8571\,\dot{4}$ we have two dots here, to show that all the figures, from the first to the last, repeat in a block.

EXERCISE 5

Convert to decimals:

a $\frac{1}{7}$ **b** $\frac{3}{7}$ **c** $\frac{4}{7}$ **d** $\frac{5}{7}$ **e** $\frac{6}{7}$

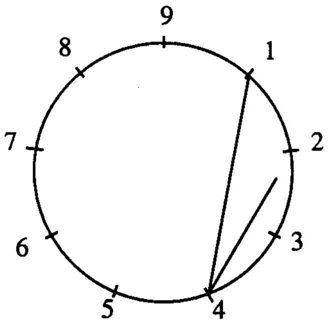
f What do you notice about the decimals for $\frac{1}{7}$ to $\frac{6}{7}$?

g $\frac{1}{11}$ **h** $\frac{2}{11}$ **i** $\frac{3}{11}$ **j** $\frac{1}{13}$ **k** $\frac{2}{13}$

l What do you notice about the decimals for $\frac{1}{11}$, $\frac{2}{11}$, $\frac{3}{11}$?

m What do you think the decimals for $\frac{4}{11}$ and $\frac{5}{11}$ will be?

Block recurrers can be plotted on the 9-point circle, each denominator having its own individual pattern or patterns.



n Draw the pattern for $\frac{1}{7} = 0.\dot{1}4285\dot{7}$ by drawing lines from 1 to 4, then 4 to 2, then 2 to 8, then 8 to 5, then 5 to 7, as shown in the sequence 142857. Then join 7 to 1 because the sequence is actually 14285714285714. . . .

o Compare this pattern with the pattern for $\frac{2}{7}$, $\frac{3}{7}$ etc.

p Draw the pattern on the 9-point circle for $\frac{1}{13}$ (remember 0 and 9 are both at the same point)

q Draw the pattern (on a separate circle) for $\frac{2}{13}$

You may have noticed with the decimals for $\frac{1}{7}$ to $\frac{6}{7}$ that the pattern is the same for each. Also the figures 14285714285714 . . . in the decimal for $\frac{1}{7}$ is the same sequence as for the other fractions, only the starting point is different.
Since 142857 has **six figures** this can account for the **six fractions** from $\frac{1}{7}$ to $\frac{6}{7}$.
However if you consider the factions with denominator 13, there are **12 fractions** from $\frac{1}{13}$ to $\frac{12}{13}$.

But the decimal for $\frac{1}{13}$ has only **6 figures**, and so can only supply 6 of the 12 possible decimals from $\frac{1}{13}$ to $\frac{12}{13}$.

This is why $\frac{2}{13}$ has a different sequence of figures in its decimal to $\frac{1}{13}$.
$\frac{2}{13}$ supplies the other 6 possible decimals.

It is possible to obtain all the twelve decimals for the fractions from $\frac{1}{13}$ to $\frac{12}{13}$ from the decimals for $\frac{1}{13}$ and $\frac{2}{13}$. You may like to see if you can do this.

RECIPROCALS

The **reciprocal** of a number is 1 divided by that number.
For example the reciprocal of 3 is $\frac{1}{3}$ which is $0.\dot{3}$

1 Copy out and complete the two centre columns of the table below showing the reciprocals of the numbers from 1 to 16. Three of the decimals have been put in for you. The others should be found by looking at the earlier examples and exercises or by doing the divisions.

Number	Reciprocal	Decimal Equivalent	Type
1	$\frac{1}{1} =$	1	
2	$\frac{1}{2} =$		
3	$\frac{1}{3} =$		
4			
5			
6			TR
7	$\frac{1}{7} =$	$0.\dot{1}4285\dot{7}$	R
8			T
9			
10	$\frac{1}{10} =$	0.1	
11			
12			
13			
14			
15			
16	$\frac{1}{16} =$		

These decimals (and all others) can now be classified as being one of three types:

T: those which terminate, like $\frac{1}{8}$

R: those which recur, like $\frac{1}{3}$ and $\frac{1}{7}$

TR: those which have a terminating part and a recurring part, like $\frac{1}{6} = 0.1\dot{6}$ (in which there is a 1 which does not recur and a 6 which does.
$\frac{1}{14}$ is also of this type.

2 Put T, R or TR in the right-hand column of your table, depending on which type it is.

You should have 7 T-types,
5 R-types,
4 TR-types.

PRIME FACTORS

If the denominator of a fraction is written as a product of prime factors we can predict whether the decimal of a fraction is a T-type, an R-type or a TR-type.

The denominators of all T-type decimals have only the factors 2 and/or 5 in them (this is because our number system is based on the number 10 which is 2X5).

In fact, in T-type decimals each 2, 5 or 10 contributes one non-recurring figure in the decimal.

for example since 4=2X2 there will be two non-recurring figures in the decimal, and since 8 = 2X2X2 there are three non-recurring figures- check with your table.

Denominators containing factors of 3, 7, 11 or higher primes (or products of these), but no 2's or 5's are R-types: they recur, either with a single figure or in a block.

For example $\frac{1}{3}$, $\frac{1}{7}$, $\frac{1}{21}$, $\frac{1}{77}$ are R-types.

Denominators which have 2's and/or 5's and 3, 7, 11 or higher primes as factors are always TR-types, with both a terminating part and a recurring part.

For example 12 = 2X2X3, so its reciprocal will have two terminating figures (because of the 2X2) and a recurring figure (because of the 3).

● Look at your table again and check that the above facts are true for these 16 fractions.

EXAMPLE 12

For $\frac{7}{24}$ (the numerator is not relevant), since $24 = 2\times2\times2\times3$ we will expect its decimal to be three figures followed by a recurring part. We can write TTTR.

EXAMPLE 13

And for $\frac{11}{42}$ since $42 = 2\times3\times7$ there will be a single figure (indicated by the 2) followed by a recurring part. This is a TR decimal.

These rules assume that the fraction is in its lowest terms:

EXAMPLE 14

For $\frac{22}{60}$ we first cancel down to $\frac{11}{30}$ and seeing a factor of 10 in the denominator (we ignore the factors of 2 and 5 if we see a factor of 10) we can predict a single terminating figure followed by a recurring part. This is also a TR decimal.

EXERCISE 6

For each of the fractions below find how many terminating figures there are in the decimal and whether it has a recurring part as well:

a $\frac{5}{18}$ **b** $\frac{9}{26}$ **c** $\frac{2}{33}$ d $\frac{29}{56}$ **e** $\frac{59}{60}$ **f** $\frac{1}{72}$

g $\frac{3}{32}$ **h** $\frac{16}{69}$ **i** $\frac{27}{46}$ **j** $\frac{3}{88}$

k $\frac{20}{34}$ (remember to cancel here) **l** $\frac{16}{150}$

The 9-point circle diagram at the beginning of each chapter of this book is the pattern for one divided by the chapter number. So for Chapter 7, for example, the recurring decimal pattern for $\frac{1}{7}$ is shown. Non-recurring decimals like $\frac{1}{5} = 0.2$ are considered to have zero recurring at the end: $0.2\dot{0}$.

4 The Arithmetics of Bar Numbers

We have studied whole, or natural numbers so far.
We now expand our number system to include integers, which are numbers that have a sign: plus or minus.
You will remember that we use bar numbers when we write, for example, $38 = 4\bar{2}$.

In this chapter we see how to add, subtract, multiply and divide bar numbers (also called negative numbers or minus numbers).

ADDITION AND SUBTRACTION

EXAMPLE 1

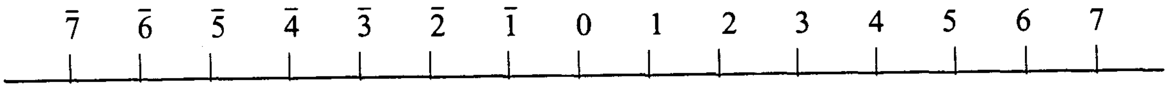
In fact we have already seen numbers like this when we solve a problem like:
If the temperature is 5° and it falls by 7° what is the new temperature?

It would clearly be 2 degrees below zero, which we call minus 2, or -2, or $\bar{2}$.

We can write this as: $5 - 7 = -2$ or $\bar{2}$.

In this kind of arithmetic every number is plus or minus. If a number has no minus sign before it or above it, then it is plus.

You may find a number line like the one below helpful with these sums. Make a copy in your book.



The Sutra in use here is the *Sankalana* Sutra: *By Addition and By Subtraction.*

EXAMPLE 2

$\bar{7} - 4 = \overline{11}$

Starting at $\bar{7}$ we go down (leftwards on the number line) 4 units and arrive at $\overline{11}$.

EXAMPLE 3

$\bar{8} + 6 = \bar{2}$

From $\bar{8}$ we go 6 units to the right, arriving at $\bar{2}$.

EXAMPLE 4

$\bar{7} + 9 = 2$

From $\bar{7}$ we go up 9 units to 2.

EXERCISE 1

Find:

a $3 - 9$ **b** $7 - 11$ **c** $4 - 15$ **d** $18 - 27$ **e** $\bar{6} - 9$ **f** $\bar{7} + 5$

g $\bar{8} + 10$ **h** $3 - 12$ **i** $\overline{11} - 17$ **j** $\overline{12} + 4$ **k** $\overline{12} + 15$ **l** $\overline{13} + 6$

☺ **GAME** for 2, 3 or 4 players.

- Each player has a board and 3 "men" and 12 cards are dealt to each player.
- Aim: to get the 3 men at START onto the squares marked 16.
- The player can look at the cards. Some are blue (plus) cards, some are red (minus) cards.
- The player to the left of the dealer lays a card of their choice and moves a piece forward by that amount. Only one piece can be moved at a time.
- When the next player lays their card it is always combined with the previous card's value. So if the first card was 15, and then $\bar{7}$ is laid, that player must move forward 8.
- So every player combines the card they lay with the one showing and moves forwards or backwards, depending on the total: for $\bar{7}$ and 2, go back 5,
  for $\bar{3}$ and $\bar{5}$ go back 8,
  and so on. ☞

- If you go beyond 27 you get to START, but you cannot go beyond START, whether going forwards or backwards: you must restart that piece.
- The game ends when all the cards have been laid.
- The winner is the first player to get their 3 men onto 16. If this is not achieved the winner is the player whose men are closest to 16 at the end.

EXAMPLE 5

We can also use the number line to add and subtract bar numbers: imagine a person who can walk along the number line. he can face in either direction and he can also walk forwards or backwards. Given $7 + \bar{6}$ for example, we start at 7 and let the minus on the 6 tell us to face in the minus direction (to the left), and let the sign (+) between the numbers tell us that the person is walking forwards.

So we start at 7 and go 6 steps to the left, arriving at 1: $7 + \bar{6} = 1$.

The first number tells us where to start.
The last number will be either positive or negative and tells us which way to face.
And we are either adding or subtracting these numbers: for adding we walk forwards
for subtracting we walk backwards.

EXAMPLE 6

$4 - \bar{5} = 9$

Here we start at 4, face the left, and step 5 units backwards. In this case we find that we actually go to the right and arrive at 9.
We can summarise this method as follows:

$$4 \overset{\text{forwards or backwards}}{-} \overset{\text{direction to face}}{\bar{7}}$$

EXERCISE 2

Find:

a $2 + \bar{4}$ **b** $3 + \bar{1}$ **c** $\bar{5} + \bar{2}$ **d** $8 + \bar{5}$ **e** $6 - \bar{2}$ **f** $\bar{4} - \bar{3}$

g $\bar{5} - \bar{7}$ **h** $10 - \overline{12}$ **i** $17 + \overline{17}$ **j** $22 + \bar{1}$ **k** $\overline{18} - \bar{5}$ **l** $3 - 4.2$

m 5 - 6½ **n** $\overline{10} + \overline{3.3}$

EXAMPLE 7

$3 \times \bar{4} = \overline{12}$

This is because $3 \times \bar{4}$ means $\bar{4}$ three times, or $\bar{4} + \bar{4} + \bar{4}$ which is $\overline{12}$.

EXERCISE 3

Find:

a $11 - 20$
b $\bar{7} + 15$
c $8 + \overline{11}$
d $\bar{9} + \bar{3}$
e $7 \times \bar{4}$
f $5 - \bar{8}$

g $\bar{6} - \overline{10}$
h $-7 - \bar{7}$
i $-19 + 9$
j $0 \times \bar{9}$
k $8 - \bar{8}$
l $5 \times \bar{3}$

APPLICATIONS IN ALGEBRA

We have seen how we can simplify various types of algebraic expressions. 14b.
For example $3a + 20b + 2(4a - 3b) = 3a + 20b + 8a - 6b =$ Check that you agree with this answer.
Now however we can tackle cases where bar numbers come in.

EXAMPLE 8

Simplify $3x + 5y - x - 7y$.

For the x terms: $3x - x = 2x$ and for the y terms: $5y - 7y = -2y$.
So the answer is $\underline{2x - 2y}$.

EXAMPLE 9

Find the value of $10 - x$ when $x = \bar{6}$.

$10 - x = 10 - \bar{6} = 16$

EXERCISE 4

Simplify the following:

a $8a - 11a$ **b** $4x - 5x$ **c** $9x + 5y - 3x - 7y$ **d** $3x - 4y -5x - 6y$

e $-7x + y -3x + 8y$ **f** $5x - 6x + 7x$ **g** $3x - 6z -3x + 3z$ **h** $2(3x + 4y) + 3(x - 3y)$

i $8(3a - 5b) - 30a -30b$ **j** $x + 4(2x - 9y) + y$ **k** $2(a + b - c) + a - 5b$

If $x = \bar{3}$ find the value of:

l $20 + x$ **m** $x - 15$ **n** $33 - x$ **o** $-x - 8$

MULTIPLICATION AND DIVISION

We can interpret the multiplication sum: $2 \times \bar{3}$ (which is a plus number multiplied by a minus number) as meaning: **do twice: subtracting 3** (i.e. subtract 3 twice).

Subtracting 3 twice means subtracting 6, so $\underline{2 \times \bar{3} = \bar{6}}$.

Next consider the case of multiplying a minus by a plus number.
We interpret $\bar{2} \times 3$ as meaning: **do the opposite twice: adding 3**.

The opposite of adding 3 is subtracting 3, so we conclude that $\underline{\bar{2} \times 3 = \bar{6}}$.

Next take $\bar{2} \times \bar{3}$ which must mean: **do the opposite twice: subtracting 3**.
This is the same as adding 6, so $\underline{\bar{2} \times \bar{3} = 6}$.

Finally the easiest case is 2×3 which means: **do twice: add 3**, which means add 6.
So $\underline{2 \times 3 = 6}$.

Summarising these four results:

If the signs of the numbers being multiplied are the same the answer is plus.
E.g. $3 \times \bar{5} = \overline{15}$,
$\bar{3} \times 5 = \overline{15}$.

If the signs are opposite the answer is minus. E.g. $3 \times 5 = 15$,
$\bar{3} \times \bar{5} = 15$.

These same rules also apply to division with bar numbers.

Since $2 \times \bar{3} = \bar{6}$ dividing both sides of this equation by 2 we get $\bar{3} = \frac{\bar{6}}{2}$ which shows that a minus number divided by a plus number gives a minus number.

Since also $\bar{2} \times \bar{3} = 6$ dividing both sides by $\bar{2}$ gives $\bar{3} = \frac{6}{\bar{2}}$ showing that a plus divided by a minus is also a minus.

Next we can divide both sides of $2 \times \bar{3} = \bar{6}$ by $\bar{3}$ we get $2 = \frac{\bar{6}}{\bar{3}}$ showing that a minus divided by a minus gives a plus, just as a plus divided by a plus gives a plus.

> For multiplication and division of bar numbers:
> Like signs give a plus,
> Opposite signs give a minus.

EXAMPLE 10

If $x = \bar{3}$ find the value of **a** $5x$ **b** $-7x$ **c** $\bar{2}x$ **d** $10 - 2x$.

a $5x = 5 \times \bar{3} = \overline{15}$ **b** $-7x = -7 \times \bar{3} = 21$ **c** $\bar{2}x = \bar{2} \times \bar{3} = 6$

d $10 - 2x = 10 - \bar{6} = 10 + 6 = 16$

EXERCISE 5

If $x = 5$ and $y = \bar{4}$ find the value of:

a $3y$ **b** $-6y$ **c** $\overline{15}y$ **d** xy **e** $-7x$ **f** $6 + 3y$

g $x - 2y$ **h** $5x + y$ **i** $-x - y$ **j** $y - 2x$ **k** $x + y - 1$

BRACKETS

EXAMPLE 11

Multiply out the brackets in **a** -3(2a - 9b) **b** 6 - (4x - 2).

a -3(2a - 9b) = -6a + 27b because -3 × 2 = -6 and -3 × -9 = +27.

b 6 - (4x - 2) = 6 - 4x + 2 = 8 - 4x here there is no number in front of the bracket, but we can suppose that there is a 1 there:
6 - (4x - 2) = 6 - 1(4x - 2)

EXERCISE 6

Multiply out the brackets and simplify where possible:

a -5(6x - 4y) **b** -8(9a + 8b) **c** -3(16a - b +53c)

d $\bar{6}$(14p - 15q) **e** -4(58x + 28) **f** -34(36t - 20)

g 4(2x - 3y) - 9(x - 5y) **h** 5(ab + 7) - 3(5ab + 10) **i** -4(a + 2b + 1) - (6a - b + 1)

NIKHILAM MULTIPLICATION AGAIN

You will recall the easy way to multiply numbers which are near to a base number, or a multiple of a base number.

EXERCISE 7

Multiply the following mentally:

a 94 × 97 **b** 88 × 95 **c** 98 × 98 **d** 89 × 89

e 104 × 109 **f** 123 × 104 **g** 1021 × 1003 **h** 888 × 997

i 987 × 995 **j** 303 × 306 **k** 71 × 72 **l** 197 × 196

So far all the examples have been where the numbers are both above or both below the base, or a multiple of the base.

EXAMPLE 12

Find 124 × 98.

Here one number is over and the other is under 100:
The differences from 100 are +24 and -2.
Cross-wise gives 122 (124-2 or 98+24).

$$\begin{array}{l} 124 + 24 \\ \underline{\ 98\ -\ \ 2} \\ \underline{122\ /\ \overline{48}} = \underline{12152} \end{array}$$

So 122 is the left-hand part of the answer.
Then multiplying the differences we get -48, written $\overline{48}$ (since a plus times a minus gives a minus). This gives the answer as $122\overline{48}$.

To remove the negative portion of the answer we just take 48 from one of the hundreds in the hundreds column. This simply means reducing the hundreds column by 1 and applying *All From 9 and the Last From 10* to 48. Thus 122 becomes 121 and $\overline{48}$ becomes 52.
So $124 \times 98 = 122\overline{48} = \underline{12152}$.

EXAMPLE 13

$1003 \times 987 = 990/\overline{039} = \underline{989/961}$

Similarly, we first get 1003-13 = 990 or 987+3 = 990,
and +3 x -13 = $\overline{039}$ (3 figures required here as the base is 1000).
Then 990 is reduced by 1 to 989, and applying the formula to 039 gives 961.

So these sums are just like the others except that we need to clear the minus part at the end.

EXAMPLE 14

$121 \times 91 = 11\overset{\bar{1}}{2}/\overline{89} = \underline{110/11}$.

Here we have a minus one to carry over to the left so that the 112 is reduced by 2 altogether.

EXERCISE 8

Find:

a 104 × 91
b 94 × 109
c 103 × 98
d 92 × 112
e 91 × 111
f 106 × 89
g 91 × 103
h 91 × 107
i 91 × 105
j 991 × 1005
k 987 × 1006
l 992 × 1111

5 Algebraic Multiplication

In this chapter we see how the algebraic notation develops to deal with algebraic multiplications, and how the numerical methods of the previous chapter can be applied in algebra,

MULTIPLYING AND DIVIDING SINGLE TERMS

It is important to remember that sometimes the number one is implied and not written.
For example where we write x we mean lx.

EXAMPLE 1

Also x means x^1 though the ^{1} is not usually written.

Find **a** $a^2 \times a^3$ **b** $b^5 \div b^2$

a Since $a^2 = a{\times}a$ and $a^3 = a{\times}a{\times}a$ then $a^2 \times a^3 = a \times a \times a \times a \times a = a^5$.

b $b^5 \div b^2 = \frac{b \times b \times b \times b \times b}{b \times b} = b^3$ since two b's cancel from the top and bottom.

These examples show that

For multiplication we add the powers, and for division we subtract the powers.

Algebraically we can write: $a^m \times a^n = a^{m+n}$ and $a^m \div a^n = a^{m-n}$.

The Sutra here is *Sankalana Vyavakalanabhyam- By Addition and By Subtraction.*

EXERCISE 1

Find:

a $a^3 \times a^5$ **b** $c \times c^3$ **c** $p^4 \times p^{10}$ **d** $a^2 \times a^4 \times a^3$ **e** $s^{15} \times s^{17}$ **f** $y \times y$

g $z^5 \times z \times z^2$ **h** $h^{29} \times h^{29}$ **i** $i^8 \div i^3$ **j** $j^7 \div j^2$ **k** $k^{50} \div k^{17}$ **l** $1^{100} \div 1$

Next we bring in the *Anurupyena* Sutra- *Proportionately.*

EXAMPLE 2

Find **a** $3x^4 \times 5x^3$ **b** $18x^6 \div 3x^2$ **c** $7x^4 \div x$.

a $3x^4 \times 5x^3 = 15x^7$ the powers are added but the coefficients are multiplied.

b $18x^6 \div 3x^2 = 6x^4$ the powers are subtracted and the coefficients are divided.

c remember $x = 1x^1$ so $7x^4 \div x = 7x^3$.

EXERCISE 2

Find:

a $5a^3 \times 3a^7$ **b** $2b^8 \times 6b^9$ **c** $c^4 \times 5c^3$ **d** $5d^5 \times 2d$

e $10e^7 \div 2e^2$ **f** $18f^9 \div 3f^5$ **g** $8g^5 \div 2g$ **h** $27h^4 \div h^3$

EXAMPLE 3

Find **a** $2a^2b \times 3a^5b^2$ **b** $9x^2y^5 \div 3xy$.

a $2a^2b \times 3a^5b^2 = 6a^7b^3$ we multiply the coefficients to get 6,
add the powers on **a** to get a^7,
add the powers on **b** to get b^3 . N.B. $\mathbf{b = b^1}$.

b $9x^2y^5 \div 3xy = 3xy^4$ divide the coefficients (9÷3=3),
subtract the powers on x and on y.

EXERCISE 3

Find:

a $x^3 \times x^2$

b $3x^7 \times 7x^2$

c $6y^3 \times y^8$

d $y^2 \times y^3 \times y^4$

e $x^{21} \times x$

f $6x \times 7x$

g $x \times 2x^2 \times 3x^3$

h $p \times p^2$

i $x^2y^3 \times x^5y^6$

j $2x^3y^5 \times 6xy^2$

k $5a^2b \times 8a^3b^2$

l $3ab \times 6a^2$

m $2a \times 3b \times 4ab$

n $8x^5 \div 2x^2$

o $42x^5 \div 6x^3$

p $18x^8 \div 6x$

q $x^9 \div x$

r $8x^5 y^{11} \div 2xy^3$

s $6a^3b^4c^2 \div 2ab^2c$

t $2x^5 \div x^3$

u $12x^7 \div 4x^2$

v $10a^5b^5 \div 2ab^2$

w $3w^3 \div w^3$

Of particular significance for some later work is the result that any number raised to the power of zero is equal to 1: $3^0 = 1$, $1.7^0 = 1$, $x^0 = 1$ etc.

This can be demonstrated as follows.
$3^2 \div 3^2 = 1$ because we are dividing a number, 9, by itself ($9 \div 9 = 1$).
But subtracting the powers as in the last exercise we get $3^2 \div 3^2 = 3^0$.
Therefore **$3^0 = 1$.**

Similarly $x^1 \div x^1 = 1$ by division and $x^1 \div x^1 = x^0$ by subtracting the powers. So **$x^0 = 1$.**

MULTIPLYING BINOMIALS

An expression with two terms is called a binomial. E.g. $2x - 3$.

EXAMPLE 4

Multiply $3x(5x - 7)$.

We have to multiply each of the terms of the binomial $(5x-7)$ by $3x$.
So $3x(5x - 7) = 15x^2 - 21x$.

Similarly, $2x(3x + 4y) = 6x^2 + 8xy$.

EXERCISE 4

Multiply the following:

a $2x(3x + 4)$

b $5x(x - 9)$

c $x(18x + 19)$

d $7y(4y - 15)$

e $3x(5x - 4y)$

EXAMPLE 5

Multiply: $(x + 3)(x + 4)$

We have to multiply $x+3$ by $x+4$.
This means that the x and the 3 in $x+3$ must both multiply the x and the 4 in $x+4$.

The best way to do this is to use the *Vertically and Cross-wise* method.

Put one binomial under the other:
Multiply vertically on the left: $x \times x = x^2$.
Cross-multiply and add: $4 \times x + 3 \times x = 7x$.
Multiply vertically on the right: $3 \times 4 = 12$.

$$\begin{array}{lr} x & +\ 3 \\ \underline{x} & \underline{+\ 4} \\ \hline \multicolumn{2}{l}{\underline{x^2 + 7x + 12}} \end{array}$$

It is just like multiplying two 2-figure numbers together.
Multiply from left to right or right to left- whichever you like.

EXERCISE 5

Multiply:

a $(x + 5)(x + 6)$
b $(x + 2)(x + 9)$
c $(x + 10)(x + 1)$
d $(x + 20)(x + 20)$
e $(x + 1)(x + 1)$
f $(x + 22)(x + 28)$
g $(y + 52)(y + 4)$
h $(x + 4)^2$

EXAMPLE 6

Multiply $(2x + 5)(3x + 2)$.

$$\begin{array}{lr} 2x & +\ 5 \\ 3x & +\ 2 \\ \hline \multicolumn{2}{l}{\underline{6x^2 + 19x + 10}} \end{array}$$

Vertically on the left: $2x \times 3x = 6x^2$.
Cross-wise: $4x + 15x = 19x$.
Vertically on the right: $5 \times 2 = 10$.

EXAMPLE 7

Multiply $(x + 3y)(5x + 7y)$.

$$\begin{array}{lr} x & +\ 3y \\ 5x & +\ 7y \\ \hline \multicolumn{2}{l}{\underline{5x^2 + 22xy + 21y^2}} \end{array}$$

On the left: $x \times 5x = 5x^2$.
Cross-wise: $7xy + 15xy = 22xy$.
On the right: $3y \times 7y = 21y^2$.

EXERCISE 6

Multiply the following:

a $(2x + 5)(x + 4)$ **b** $(x + 8)(3x + 11)$ **c** $(2x + 1)(2x + 20)$ **d** $(2x + 3)(3x + 7)$

e $(4x + 3)(x + 6)$ **f** $(3x + 17)(3x + 4)$ **g** $(6x + 1)(5x + 1)$ **h** $(2x + 5)(4x + 5)$

i $(3x + 3)(4x + 5)$ **j** $(2x + 3y)(2x + 5y)$ **k** $(5x + 2y)(2x + 5y)$ **l** $(4x + 3y)(7x + y)$

m $(7x + y)(x + 7y)$ **n** $(x + y)(x + y)$

Next we need to use the methods for combining negative numbers.

EXAMPLE 8

Multiply $(2x - 3)(3x + 4)$.

This is very similar:
$2x \times 3x = 6x^2$,
Cross-wise: $8x - 9x = -1x$ or $-x$.
And $-3 \times 4 = -12$.

$$\begin{array}{lr} 2x & -\ 3 \\ 3x & +\ 4 \\ \hline 6x^2\ \ -x & -\ 12 \\ \hline \end{array}$$

EXAMPLE 9

Find $(x - 3)(x - 6)$.

Vertically: $x \times x = x^2$,
Cross-wise: $-6x - 3x = -9x$,
Vertically: $-3 \times -6 = +18$.

$$\begin{array}{lcr} x & - & 3 \\ x & - & 6 \\ \hline x^2 - 9x + 18 \\ \hline \end{array}$$

EXERCISE 7

Multiply:

a $(x + 3)(x - 5)$ **b** $(x + 7)(x - 2)$ **d** $(x - 4)(x + 5)$ **d** $(x - 5)(x - 4)$

e $(x - 3)(x - 3)$ **f** $(2x - 3)(x + 4)$ **g** $(2x - 3)(3x + 6)$ **h** $(3x - 1)(x + 7)$

i $(4x + 3)(2x - 5)$ **j** $(x + 5)(4x - 7)$ **k** $(x + 1)(9x - 1)$ **l** $(2x + 1)(2x - 1)$

There may be more than one letter in the sum:

EXAMPLE 10

Multiply (x - y)(2x - 3y).

$x \times 2x = 2x^2$,
cross-wise: $-3xy - 2xy = -5xy$,
$-y \times -3y = +3y^2$.

$$\begin{array}{lcr} x & & -\ y \\ \underline{2x} & & \underline{-\ 3y} \\ \underline{2x^2} & \underline{-\ 5xy} & \underline{+\ 3y^2} \end{array}$$

EXAMPLE 11

Multiply (a + b)(c + d).

$$\begin{array}{lcr} a & + & b \\ \underline{c} & \underline{+} & \underline{d} \\ \underline{ac + ad} & \underline{+ bc} & \underline{+ bd} \end{array}$$

EXERCISE 8

Multiply:

a $(3x + 4y)(2x - 3y)$
b $(6x + y)(2x - 5y)$
c $(x + 2y)(2x - y)$
d $(3x - 4y)(2x + 3y)$
e $(3x - 4y)(2x - 3y)$
f $(4x - 5y)(4x + 5y)$
g $2x(3x - 4y)$
h $6x(x + 8y)$
i $a(b + c)$

FACTORISING QUADRATIC EXPRESSIONS

The reverse process is also important.
We need to be able, given an expression like $2x^2 - 6x$, to write it as a product.

EXAMPLE 12

Factorise $2x^2 - 6x$.

In the two terms here we see a common factor of 2 and also a common factor of x.
We can therefore take out 2x as a factor and write: $2x^2 - 6x = \underline{2x(x - 3)}$.

We can check that this is correct by multiplying out $2x(x - 3)$ to get $2x^2 - 6x$.

Similarly, given $3x^2 - 6$, only 3 is a common factor so $3x^2 - 6 = \underline{3(x^2 - 2)}$.

EXERCISE 9

Factorise the following:

a $12x^2 + 9x$

b $8x^2 - 10x$

c $4x^2 - 7x$

d $x^2 + 9x$

e $10x^2 + 15x$

f $10x^2 + 15$

g $9y^2 - 5y$

h $g^2 + 19g$

i $2x^2 - x$

j $6x^2 + 9x$

k $6x^2 + 8x$

l $6x^2 + 8$

m $6x^2 + 7x$

n $6x^2 - 6x$

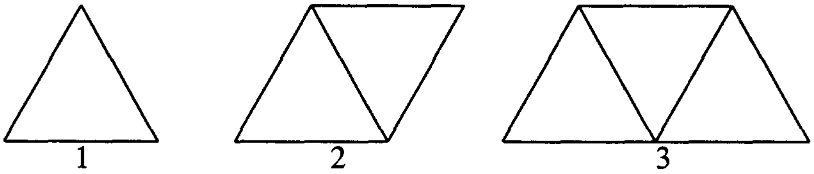
o $6x^2 - 6$

Factorising quadratic expressions consisting of three terms will be dealt with later.

6 Sequences

In this chapter, we will be looking at some simple number sequences and generating formulae for these patterns.
Some of these sequences can be built up using simple geometric shapes and patterns. We are going to start with the pattern below.

1 Make the shapes shown below with matchsticks and then fill in the table below, by extending the pattern.



Number of shape	1	2	3	4	5	6
Number of matchsticks	3					

The Sutras in use in this chapter are *By One More than the One Before* and *By Mere Observation.*

Be careful not to confuse this pattern with the triangular number sequence in Chapter 6 of Book 1. You may need to look back at that chapter to see the difference. This is a sequence that happens to be generated out of triangles!
You have seen sequences like this before and you have had practice finding the next one or two terms. In Chapter 6 of Book 1 you were asked to find the 30th term.

- Check that you agree the next term in the table above would be 15 and the 30th term 61.

This is all very well, but when you get on to high numbers like this, it is a little awkward to have to count all the way through every possible number until you arrive at the one you want. What would you do if you wanted the 86th term? It would be much more convenient to have a general formula into which we could substitute the number we wanted, say the 7th, 20th or even 50th term and produce the answer straight away. This is exactly what we are going to try and do, by producing what is usually called a formula for the nth term.

THE Nth TERM

Now look at the basic sequence: 1, 2, 3, 4, 5, 6 ....

This sequence, which goes up by 1 each time, is special because each number gives the position of that number: 3 is in the 3rd place, 7 is in the 7th place. And n is in the nth place.

This is very simple but it means that we can describe other sequences in terms of n using this basic sequence.

For example, in the sequence 2 , 4 , 6 , 8 , 10 , 12 , 14 . . . . the numbers are all double those in the basic sequence 1, 2, 3, 4, 5, 6, 7 ...
The numbers also go up by 2 each time:

$$\mathbf{2} \xrightarrow[+2]{,} \mathbf{4} \xrightarrow[+2]{,} \mathbf{6} \xrightarrow[+2]{,} \mathbf{8} \xrightarrow[+2]{,} \mathbf{10} \xrightarrow[+2]{,} \mathbf{12} \xrightarrow[+2]{,} \mathbf{14} \; \cdots\cdot$$

This gives us a very important clue as to the general formula or nth term. This means the nth term for this sequence is given by the formula 2n.

The following sequence goes up by 3 each time and each number is 3 times the basic sequence 1,2,3,4,5,6,7...

$$\mathbf{3} \xrightarrow[+3]{,} \mathbf{6} \xrightarrow[+3]{,} \mathbf{9} \xrightarrow[+3]{,} \mathbf{12} \xrightarrow[+3]{,} \mathbf{15} \xrightarrow[+3]{,} \mathbf{18} \xrightarrow[+3]{,} \mathbf{21} \; \cdots\cdot$$

The nth term here would be given by 3n.

2 What do you think the nth term is for the following sequence?

5,10,15,20,25,30....

3 What do you think the nth term is for the following sequence?

6 ,12,18,24,30,36....

4 What do you think the nth term is for the following sequence?

12,24,36,48,60,72....

Now let us return to the sequence in your table: 3 , 5 , 7 , 9 ,11 , 13 , 15.....

We notice that the sequence goes up by 2 each time, so the formula will be similar to 2n. What the difference between this sequence and the one for which the nth term is (2,4,6,8,10,12,14.....)?

The numbers are all one more than in this sequence. so the nth term here is 2n+1.

We can also check that we can produce the correct sequence from the formula as follows:

Using the formula 2n+1,

If n=1, 2n+1 = **3**,
if n=2, 2n+1 = **5**,
if n=3, 2n+1 = **7**,
if n=4, 2n+1 = **9**,
if n=5, 2n+1 = **11** etc.

This is exactly the sequence we started with. We can also relate this to our original pattern with equilateral triangles. For each additional triangle we made we needed to add two extra matchsticks, except for the first one where we started with three (one more than two). This original triangle or base triangle is then always there, no matter how many extra triangles we add. This equates clearly with the formula 2n+1.

EXAMPLE 1

Find the nth term for the following sequence:

8 , 11 , 14 ,17 , 20 , 23....

The increase each time is 3, so the formula will contain **3n**.
But **3n** generates **3, 6, 9, 12, 15 . . . .** and in the sequence above each number is **5 more** thai this.
So the formula will be **3n+5**.

Check: putting n=1,2,3,4,5... into this formula gives us the given series: 8,11,14,17,20.....

Finding the nth term of a sequence that increases by a regular amount each time is summarised below.

a) First determine the amount you go up by each time. This will tell you whether it is 2n, 3n, 4n etc. For example, if it increases by 5 each time it will be 5n.
b) Look at the first term of the given sequence. If it is more than the amount by which the sequence increases, add the extra amount to your n term.
If the first term is less than the amount by which the sequence increases, subtract from your n term as shown in the next example.
c) Check that you can generate the sequence you require from your formula.

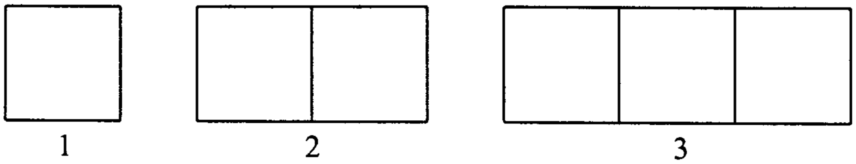
EXAMPLE 2

Find the nth term for the following sequence:

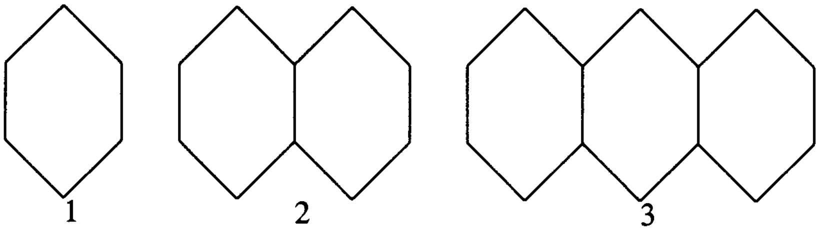
1 , 8 , 15 , 22 , 29 , 36....

The formula starts with 7n, as it increases by 7 each time.
We started at 1 not 7. 1 is 6 below 7, so the formula must be **7n-6**
Putting n=1,2,3,4,5... into 7n-6 gives us 1,8,15,22,29....

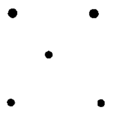
5 Use matches to make the following shapes and draw a table similar to that on page 20. Then generate a formula for the nth term. Check you can generate the sequence from your formula.



6 Do the same thing for the following pattern as you did for question 4.



7 This time continue the following pattern in your book, make a table and find a formula. Check your formula.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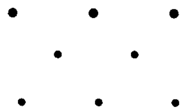
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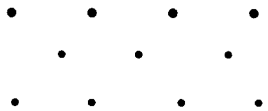

3

EXERCISE 1

Find a formula for the nth term of the following sequences. Check your formulae mentally.

a 8,16,24,32,40,48.....
b 10,20,30,40,50,60.....
c 1,2,3,4,5.6.....
d 3,4,5,6,7,8....
e 5,7,9,11,13,15...
f 2,5,8,11,14,17.....
g 12,18,24,30,36,42.....
h 15,20,25,30,35,40.....
i 2,7,12,17,22,27.....
j 3,12,21,30,39,48.....
k 0,4,8,12,16,20
l 72,75,78,81,84,87.....

EXERCISE 2

Go back to Exercise 1 and by using the formulae you have found write down the 30th, 60th and 87th term in each sequence.

SEQUENCES INVOLVING FRACTIONS

EXAMPLE 3

It may be that we want to find the formula for a series involving fractions.

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \cdots\cdot$$

Here it is best to consider the numerator and denominator separately. The numerator does not really have a formula, it is always 1. The formula for the denominator is n+1. So the nth term would be given by $\frac{1}{n+1}$.

EXAMPLE 4

What is the nth term of the following sequence?

$$\frac{3}{2}, \frac{5}{6}, \frac{7}{10}, \frac{9}{14}, \frac{11}{18} \cdots\cdot$$

The formula for the numerator is 2n+1 and the formula for the denominator is 4n-2. So the nth term would be $\frac{2n+1}{4n-2}$.

EXERCISE 3

Find formulae for the following sequences. Again check your answers mentally.

a $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots.$

b $\frac{1}{4}, \frac{1}{8}, \frac{1}{12}, \frac{1}{16}, \frac{1}{20} \ldots.$

c $\frac{1}{1}, \frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \frac{1}{13} \ldots.$

d $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6} \ldots.$

e $\frac{1}{6}, \frac{2}{11}, \frac{3}{16}, \frac{4}{21}, \frac{5}{26} \ldots.$

f $\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \frac{15}{20} \ldots.$

g $\frac{2}{3}, \frac{5}{5}, \frac{8}{7}, \frac{11}{9}, \frac{14}{11} \ldots.$

h $\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{5}{5} \ldots.$

What does the last sequence in part h simplify to?

7 Probability

CERTAIN, UNCERTAIN AND IMPOSSIBLE

Some things are certain to happen, for example it is certain that the number following an even number will be odd.
Some things are impossible, for example finding a triangle with four sides.
These are two extremes and in between are things that may or may not happen. For example, it may or it may not rain tomorrow.

impossible certain

This line shows the range of possibilities, from impossible to certain. Where, on this line, would you put the possibility of rain tomorrow?

EXERCISE 1

For each of the following decide how likely they are, and put the letters a,b,c ....j on a copy of the above line, where you think they should go:

a Half the class will be away tomorrow.

b It will snow here in December.

c It will snow here in April.

d It will snow here in August.

e The next car you see will be red.

f It will rain here this week.

☞

g You will meet your favourite pop star today.

h It will be exactly 4 o'clock when you get home.

i The next person you meet will have a name beginning with J.

j If you toss a coin it will be heads.

☺ **A GAME** for 2 to 8 players.

You will need the board, a counter for each player, a coin and a dice.
About half the players are in the coins team and half in the dice team.
The coins team start with their counters at one end of the board and the dice team at the other end.
Each player chooses a lane.
The aim is for one team to get all their counters to the other end before the other team.
Toss the coin to decide which team starts, then take turns: the coins team always tossing their coin and the dice team always throwing their dice.
Each player in the coins team moves their counters forward 6 places if it is a head and 3 places if it is a tail.
The dice team move forward by the number shown on the dice.
Play the game at least four times.

1 Are the coins team or the dice team more likely to win?

2 How can you find out which team has the better chance of winning?

3 How could you change the scores for heads and tails so that each team has an equal chance of winning?

SCALE OF PROBABILITIES

It is usual for the scale of probabilities to go from 0 to 1.

0____1
impossible certain

So impossible events have a probability of 0.
And certain events have a probability of 1.
All other events have a probability value between 0 and 1.
And events which are equally likely to happen or not happen have a probability of ½ or 0.5 (sometimes called "evens").

4 Go back to the list in Exercise 1 and give a value (between 0 and 1) for each event.
If the answer is not obvious describe what you could do, or what you would need to know, to get some idea of the answer.

POSSIBLE OUTCOMES

It is often useful when studying probabilities to know what the possible outcomes of an event are.

For example the possible outcomes of a football match are win, lose, draw: 3 possibilities.

For a person's star sign there are 12 possibilities: Aries, Gemini etc.

EXERCISE 2

For each of the following give the number of possible outcomes:

a a coin is tossed

b a die is thrown

c the first letter of someone's name

d the day of the week of someone's birthday

e the day of the year of someone's birthday

f the first letter of the month of someone's birthday

5 We said there are three possibilities for the outcome of a football match: win, lose or draw. Do you think the probabilities for these are $\frac{1}{3}$, $\frac{1}{3}$ and $\frac{1}{3}$? If not, why not?

6 How could you estimate the probabilities for a particular football team?

7 How you could estimate the probability of rain next May 1st?

EXAMPLE 1

The outcome of the last 12 matches of a particular football team is known to be:

W W L D W L L W D W W L where W=win, L=lose and D=draw.

Use this information to estimate what is most likely and least likely to happen in their next match, and find out the probabilities for win, lose and draw.

We can begin by counting up how many times they have won, lost and drawn:

there are 6 wins,
4 matches lost
and 2 drawn.

Clearly **winning is most likely and drawing is least likely in the next match**, based on the last 12 matches.

☞

Since they have won 6 times out of 12 the probability of winning will be $\frac{6}{12}$ or $\frac{1}{2}$.
Similarly they lost 4 times so the probability of losing is $\frac{4}{12} = \frac{1}{3}$.
And 2 draws means a probability of $\frac{2}{12} = \frac{1}{6}$.

The probability of an event is given by a fraction:

$$\text{Probability} = \frac{\text{number of times the event has happened}}{\text{maximum number of times it could have happened}}$$

EXERCISE 3

a The outcome of the previous 10 matches for a team are: L D W L L D D W L W.
Find the probability of a win, lose and draw. What is the most likely outcome of the next match, based on this information?

b You have the following information about another team (team B) who are going to play team A above: DWLLDWWLLL.
Find the probabilities for team B. Which team do you think is more likely to win the match?

c The following information is available about teams X and Y who are going to play each other:
Team X: WLWWDWWLLWWD
Team Y: WDWWLW
Find the probabilities of win, lose and draw for both teams and decide who is favourite to win.

d Make a list of other factors that may need to be considered before deciding (e.g. players who are injured).
If you were not given the information about the football teams above you might have to find it yourself or find some other way of choosing between them (for example you could ask a number of people what they thought would happen).
If you need to know the probability of a car being red you could try to find out how many red cars there are and how many of other colours.

This could involve you in a survey to find out the numbers of different colours of car passing a particular place at a particular time. The probabilities could then be found from this information.

But there is another way of finding probabilities which does not rely on previous results or a survey.

THEORETICAL PROBABILITIES

If we toss a fair coin we know that we have 1 chance in 2 of getting heads so the probability of getting heads is $\frac{1}{2}$.

EXAMPLE 2

Find the probability of getting a 5 when throwing a fair dice.

There are 6 possible outcomes and only 1 of them is a 5,
so the probability is 1 in 6 or $\frac{1}{6}$.

EXAMPLE 3

Find the probability of getting an odd number when throwing a fair dice. There are 3 possibilities: 1, 3 or 5 so the probability is 3/6 = 1/2 .

Theoretical probability is given by a fraction:

$$\text{Probability} = \frac{\text{number of ways in which an event can ocurr}}{\text{total number of possible outcomes}}$$

EXAMPLE 4

Find the probability of picking a vowel out of a bag containing all the letters of the alphabet.

There are 26 letters altogether and 5 of them are vowels,
so the probability is $\frac{5}{26}$.

EXERCISE 4

Find the probability of (cancel fractions where possible):

a getting 5 or 6 when throwing a fair dice

b picking the letter A from a bag containing all the vowels c throwing a number less than 3 with a fair dice

d picking a club from a pack of cards e picking an ace from a pack of cards

f getting a number below 7 when throwing a fair dice.

g 3 or less when two dice are thrown and the score is the total on the two dice.

8 Equations

Another use of algebra is when we are given an equation which contains a letter, and we are asked to find the value that the letter represents.

ONE-STEP EQUATIONS

If, for example, we are given the equation $x + 5 = 8$ we need to know what x can be so that when 5 is added to it, the total is 8.
This is clearly 3 because $3 + 5 = 8$. So we say

EXAMPLE 1

Solve the equation $9 + x = 20$.

Again it is not difficult to see that $\underline{x=11}$, because $9+11=20$.

You may have noticed in these two examples that you can get the answer by taking the number next to the x from the number on the other side of the equals sign.

EXAMPLE 2

Solve $x + 39 = 70$.

So to solve this equation we just take 39 from 70, so $\underline{x=31}$.

EXAMPLE 3

Solve $x - 7 = 8$.

This says that when 7 is subtracted from a number we get 8.
What is the number?
We get $\underline{x=15}$, since $15-7=8$.

Notice again that we can get the answer easily, this time by adding the 7 to the 8.

EXAMPLE 4

Solve $x - 13 = 30$.

We find $x = 43$.

So for equations like $x + 6 = 7$ or $6 + x = 7$ we take the number on the left from the number on the right.

And for equations like $x - 6 = 7$ we add the number on the left to the number on the right.

The Vedic Sutra operating here is *Paravartya Yojayet* which means *Transpose and Apply.*

Transpose means "reverse" and in solving equations *Transpose and Apply* means

> where something is **added** to the x-term: **subtract**,
> where something is **subtracted** from the x-term: **add**.

EXERCISE 1

Solve the following equations, check each answer to make sure it is right:

a $x + 3 = 10$
b $x - 3 = 10$
c $x + 4 = 11$
d $20 + x = 100$
e $x - 6 = 2$
f $x - 15 = 7$
g $x - 19 = 44$
h $x + 88 = 100$
i $x - 3\frac{1}{2} = 4\frac{1}{2}$
j $x + 16 = 60$
k $x + 123 = 1000$
l $x - 18 = 18$
m $x + 1.3 = 5$

EXAMPLE 5

Solve $3x = 15$

This says that 3 times a number is 15, so the answer is clearly 5, as 3 fives make 15.
So we write $x=5$.

EXAMPLE 6

Solve $7x = 28$.

$x=4$ (since 7 fours make 28).

Again you may notice that the easy way to get the answer is to divide the number on the right by the number on the left: 15÷3=5 and 28÷7=4.
So the *Transpose and Apply* formula is working here too: where x is **multiplied** by a number we **divide**. Divide is the opposite of multiply.

And if x is divided by a number we would expect to multiply:

EXAMPLE 7

Solve $\frac{x}{3} = 7$.

So x must be 3×7. x=21
We can see this is right because $\frac{21}{3} = 7$.

EXAMPLE 8

$\frac{x}{23} = 30$. Since 23×30=690 we can say x=690.

EXERCISE 2

Solve the following, checking your answer each time:

a $3x = 21$ **b** $5x = 35$ **c** $2x = 26$ **d** $4x = 36$ **e** $6x = 54$

f $3x = 960$ **g** $2x = 76$ **h** $40x = 120$ **i** $3x = 333$ **j** $7x = 98$

k $2\frac{1}{2}x=10$ **l** $3\frac{1}{2}x=21$ **m** $\frac{x}{4}=5$ **n** $\frac{x}{3}=8$ **o** $\frac{x}{13}=3$

p $\frac{x}{3}=19$ **q** $2x=7$ **r** $\frac{x}{10}=40$ **s** $\frac{x}{60}=60$ **t** $6x=1800$

u $2x = 3.6$

TWO-STEP EQUATIONS

We have seen how the *Transpose and Apply* formula can be used in solving equations.

Sometimes two or more applications of the formula are needed, as the following examples show.

EXAMPLE 9

Solve $2x + 3 = 13$.

Can you see what x is here? A number is doubled and three is added and the result is 13.

You can first take 3 from both sides of the equation: this gives $2x = 10$.

Then you can see that $\underline{x=5}$ is the answer.

To check: $2\times\underline{5} + 3 = 13$ so it is correct.
There are two applications of *Transpose and Apply* here:
First the $+3$ indicates that we subtract 3 from 13 (to get 10),
then the $2x$ indicates that we divide 10 by 2.

EXAMPLE 10

Solve $5x - 4 = 36$.

Using the Sutra we add 4 to 36 to get 40,
then $40\div5 = 8$, so $\underline{x=8}$.

Check: $5\times8 - 4 = 36$.

If you like you can write the sum out in steps like this: $5x - 4 = 36$
$5x = 40$
$\underline{\quad x=8}$

But you should also be able to put the answer straight down.

EXAMPLE 11

Solve $\frac{x}{7} + 3 = 5$.

Here we take 3 from 5 to get 2,
then multiply 2 by 7, so $\underline{x=14}$.

EXAMPLE 12

Solve $\frac{2x}{3} = 4$.

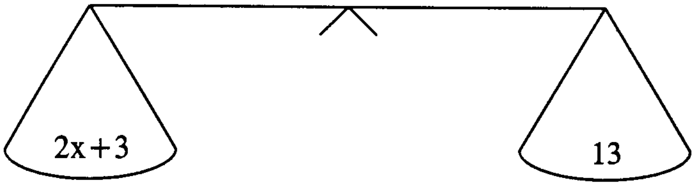
Multiply 3 by 4 to get 12,
then $12\div2=6$, so $\underline{x=6}$.

EXAMPLE 13

Solve $\frac{x-3}{4} = 5$.

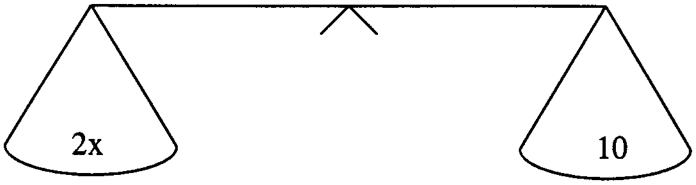
Because all the left side is divided by 4 we begin by multiplying 5 by 4, then we add 3 to the result giving $\underline{x = 23}$.

It is useful to think of a pair of scales when solving equations:



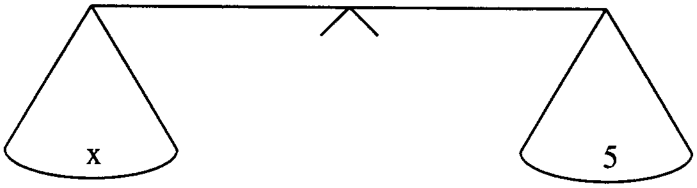
$\underline{2x + 3 = 1}$

Because the two sides of the equation are equal, they balance.
And they will still balance if we add the same amount to each side, or subtract the same amount from each side:



take 3 from both sides
to get **$\underline{2x = 10}$**

And they will still balance if we double each side, or multiply both sides by any number, or divide both sides by any number:



halve both sides
to get **$\underline{x = 5}$**

This is a very useful way of dealing with equations.

EXERCISE 3

Solve the following equations mentally. Check your answers.

a $3x + 7 = 19$
b $2x + 11 = 21$
c $3x + 5 = 29$
d $7x + 10 = 31$
e $4x - 5 = 7$
f $3x - 8 = 10$
g $5x - 21 = 4$
h $2x - 5 = 6$
i $\frac{x}{3} + 4 = 6$
j $\frac{x}{4} + 7 = 9$
k $\frac{x}{2} - 8 = 2$
l $\frac{x}{3} - 1 = 6$
m $\frac{2x}{3} = 8$
n $\frac{3x}{4} = 15$
o $\frac{5x}{3} = 15$
p $\frac{2x}{5} = 20$
q $\frac{x+4}{7} = 5$
r $\frac{x-21}{10} = 1$
s $2x + 1 = 3.8$
t $3x + 2 = 6.11$

THREE- STEP EQUATIONS

EXAMPLE 14

Solve $\frac{3x}{5} + 4 = 10$.

First $10 - 4 = 6$, then $6 \times 5 = 30$, then $30 \div 3 = 10$ so $\underline{x = 10}$.

EXAMPLE 15

Solve $\frac{3x+2}{4} = 8$.

First $8 \times 4 = 32$, then $32 - 2 = 30$, then $30 \div 3 = 10$ so $\underline{x = 10}$.

EXAMPLE 16

Solve $2(3x + 4) = 38$.

The bracket here indicates that 3x+4 is being multiplied by the number outside the bracket, which is 2.
So we begin by dividing 38 by 2.

First $38 \div 2 = 19$, then $19 - 4 = 15$, then $15 \div 3 = 5$ so $\underline{x = 5}$.

Alternatively, here, we can multiply the bracket out first:
If $2(3x + 4) = 38$ then $6x + 8 = 38$
and so $38 - 8 = 30$ and $30 \div 6 = \underline{5}$.

- Show that $x=5$ is the solution to $2(3x + 4) = 38$.

EXERCISE 4

Solve the following:

a $\frac{2x}{3} + 4 = 8$ **b** $\frac{3x}{5} - 4 = 5$ **c** $\frac{7x}{2} - 10 = 11$ **d** $\frac{3x}{8} + 17 = 20$

e $\frac{2x+1}{3} = 4$ **f** $\frac{2x-3}{5} = 3$ **g** $\frac{5x+2}{3} = 9$ **h** $\frac{6x-1}{7} = 5$

i $3(5x - 2) = 54$ **j** $8(x + 3) = 64$ **k** $3(7x - 3) = 33$ **l** $2(4x + 3) = 102$

We sometimes have to work with bar numbers when solving equations as the next examples show.

EXAMPLE 17

Solve **a** $3x = \overline{15}$ **b** $-7x = \overline{56}$ **c** $\bar{2}x + 5 = 17$ **d** $3x + 11 = 2.$

a $x = \frac{\overline{15}}{3} = \bar{5}$ **b** $x = \frac{\overline{56}}{\bar{7}} = 8$ **c** $\bar{2}x = 12,\ x = \frac{12}{\bar{2}} = \bar{6}$

d $3x = 2 - 11,$
$3x = -9,$
$x = -3.$

EXAMPLE 18

Solve $\bar{3}x = 24.$

We get $x = \frac{24}{\bar{3}}$ so $x = -8$ or $\bar{8}.$

EXERCISE 5

Solve:

a $5x = \overline{55}$ **b** $\bar{4}x = 48$ **c** $-3x = \overline{90}$ **d** $5x + 21 = 1$

e $8x - 2 = \overline{26}$ **f** $2x = \bar{7}$ **g** $3x = \overline{3.9}$ **h** $\bar{7}x - 3 = 4$

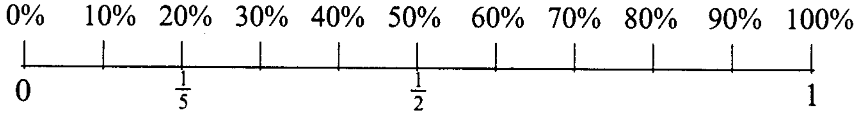
i $\bar{5}x = 20$ **j** $\bar{9}x = \overline{54}$ **k** $-3x - 7 = \overline{10}$ **l** $3x = \overline{6.6}$

m $2x = \overline{1.2}$

9 Percentages

A percentage is simply a fraction in which the denominator is 100.

So 20% (20 per cent) means $\frac{20}{100}$. And this cancels down to $\frac{1}{5}$.
Similarly 50% means $\frac{1}{2}$, and 100% means a whole one.



CONVERTING A PERCENTAGE TO A FRACTION

It is therefore very easy to convert percentages to fractions: we simply write the number over 100 and cancel if possible.

EXAMPLE 1

Change 15% to a fraction.

$15\% = \frac{15}{100} = \frac{3}{20}$.

We change a percentage to a fraction by dividing the number by 100.

EXAMPLE 2

Change 37½% to a fraction.

$37\frac{1}{2}\% = \frac{37\frac{1}{2}}{100} = \frac{75}{200} = \frac{3}{8}$.

Here we double the top and bottom of the fraction $\frac{37\frac{1}{2}}{100}$ to get $\frac{75}{200}$ and then cancel by 25.

EXERCISE 1

Convert the following percentages to fractions:

a 40%
b 90%
c 25%
d 50%
e 75%
f 35%
g 5%
h 10%
i 56%
j 17%
k 12½%
l 62½%
m $33\frac{1}{3}\%$

CONVERTING A FRACTION TO A PERCENTAGE

The reverse process is also quite easy.
We simply multiply top and bottom of the fraction (if necessary) so that we get 100 in the bottom.

EXAMPLE 3

Convert **a** $\frac{15}{100}$ **b** $\frac{8}{50}$ **c** $\frac{7}{20}$ **d** $\frac{4}{5}$ to percentages.

a $\frac{15}{100} = \underline{15\%}$

b $\frac{8}{50} = \frac{16}{100} = \underline{16\%}$

c $\frac{7}{20} = \frac{35}{100} = \underline{35\%}$

d $\frac{4}{5} = \frac{80}{100} = \underline{80\%}$

EXAMPLE 4

Convert $\frac{5}{6}$ to a percentage.

Here we cannot find a whole number to multiply 6 by to get 100.
We therefore multiply the fraction by 100 (which is just the reverse of the previous method of converting a fraction to a percentage).

$\frac{5}{6} \times 100\% = \frac{500}{6}\% = \frac{250}{3}\% = 83\frac{1}{3}\%$.

We could also have used this method in Example 3: $\frac{7}{20} = \frac{7}{20} \times 100\% = \frac{700}{20}\% = 35\%$.

EXERCISE 2

Change the following to percentages:

a $\frac{85}{100}$ **b** $\frac{3}{20}$ **c** $\frac{1}{10}$ **d** $\frac{3}{4}$ **e** $\frac{24}{25}$ **f** $\frac{27}{50}$ **g** $\frac{2}{9}$ **h** $\frac{2}{3}$ **i** $\frac{1}{7}$

IMPORTANT PERCENTAGES

The following percentage equivalents are worth knowing as they arise quite frequently:

$$10\% = \frac{1}{10} \quad 20\% = \frac{1}{5} \quad 25\% = \frac{1}{4} \quad 50\% = \frac{1}{2} \quad 75\% = \frac{3}{4} \quad 12\frac{1}{2}\% = \frac{1}{8}$$

Many percentage questions can be answered easily and mentally using these basic equivalents.

CONVERTING BETWEEN PERCENTAGES, FRACTIONS AND DECIMALS

Converting between percentages and decimals is an easy matter as the denominator of the percentage expressed as a fraction is always 100.

EXAMPLE 5

Change 65% to a decimal.

$65 \div 100 = \underline{0.65}$. (because 65% means $\frac{65}{100}$)

EXAMPLE 6

Change 0.3 to a percentage.

Here we multiply by 100: $0.3 \times 100 = 30$.
So 0.3 = 30%

10% is a particularly useful percentage as it is $\frac{1}{10}$ as a fraction and 0.1 as a decimal.

EXERCISE 3

Copy and complete the following table (the first one has been done for you):

Percentage	Fraction	Decimal
70	$\frac{7}{10}$	0.7
15		
52		
1		
	$\frac{1}{4}$	
	$\frac{2}{5}$	
	$\frac{11}{20}$	
		0.12
		0.5
		0.13

All this comes under the Vedic formula *Proportionately.*

FINDING A PERCENTAGE OF A QUANTITY

We sometimes need to find a percentage of something, if for example we are offered a 15% discount on something we want to buy.

EXAMPLE 7

Find 20% of 70.

We can see mentally that 10% of 70 is 7, so 20% must be 14.

EXAMPLE 8

Find 15% of 60.

Since 10% of 60 is 6, 5% will be 3. Therefore 15% must be 6 + 3 = 9.

EXAMPLE 9

Find 7½% of 80.

Here we spot that 7½ is 3×2½. And 2½ is a quarter of 10.
10% will be 8, so 5% will be 4 and 2½% will be 12.
So 7½% of 80 will be 3×12 = 36.

EXAMPLE 10

Find 3% of 5000.

Here we can find 1% of 5000, which is 50, then 3% must be 3×50 = 150.

EXERCISE 4

Find:

a 10% of 90
b 20% of 50
c 30% of 300
d 90% of 40
e 80% of 2000
f 25% of 60
g 75% of 28
h 15% of 40
i 2½% of 80
j 70% of 500
k 80% of 20
l 20% of 80
m 45% of 60
n 7½% of 200
o 3% of 700
p 23% of 100
q 13% of 2000
r 12½% of 200
s 6% of 80,000
t 55% of 60

EXAMPLE 11

Find 7% of 3.

The best way here is to find 1% of 3 and multiply by 7.
1% of 3 is 0.03, so 7% is 7×0.03 = 0.21.

EXAMPLE 12

Find 37% of 20.

Again we find 1% first: 1% of 20 is 0.2, so 37% is 37×0.2 = 7.4.

EXAMPLE 13

Find 34% of 36.

1% of 36 is 0.36, so 34% is 34×0.36 and we recognise 34×36 as being easy under the formula *By One More than the One Before.*
So 34% of 36 = 12.24.

EXERCISE 5

Find:

a 3% of 9
b 11% of 4
c 12% of 20
d 28% of 2
e 33% of 3
f 13% of 6
g 8% of 21
h 6% of 33
i 3% of 88
j 17% of 4
k 6% of 53
l 36% of 2
m 31% of 4
n 23% of 5

o 600 pupils sit an examination. 85% passed and 8% gained distinctions. How many passed the exam and how many obtained a distinction?

p The area of a field is 5 acres. If 20% of the field is unusable how many acres can be used?

You may have noticed in Exercise 4, k and 1 that the answers to 80% of 20 and 20% of 80 are the same. This is generally true, so if we had for example 37% of 20 we could use *Transpose and Apply* and find 20% of 37, knowing that it will give the same answer.

FORMING A PERCENTAGE

Sometimes we need to form our own percentages.

EXAMPLE 14

Jane got 15 marks out of 20 in a test. What percentage mark did she get?

It may be obvious from the question that Jane got three quarters of the marks.
And this is equivalent to 75%.

We can also write 15 out of 20 as a fraction, $\frac{15}{20}$, and convert this to a decimal by multiplying top and bottom by 5. This gives 75%.

To find what percentage one thing is of another we form a fraction (the first over the second) and convert this to a percentage.

EXAMPLE 15

What percentage is 20p of £3?

We must not mix the units so we will decide to work in pence:

$\frac{20}{300} \times 100\% = \frac{20}{3}\%$ (cancelling by 100),

EXERCISE 6

Find what percentage the first quantity is of the second:

a 2 of 8	**b** 9 of 12	**c** 20 of 400	**d** 6 of 15
e £1.20 of £5	**f** 20 of 1000	**g** 4cm of 1m	**h** 45g of 75g

i What percentage of 20 is 8?

j What percentage of £1 is 10p?

k Alan got 56 marks out of 80 in a test. In the next test he got 45 marks out of 60. Find the percentage mark in each test. Did Alan do better in the second test or worse?

l 15cm is sawn off a board 120cm long. What percentage of the board has been removed? What percentage is left?

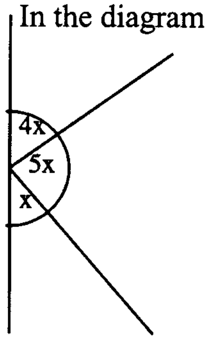
● Find out what percentage of your class are left-handed. ~ Find out what percentage of your class have blue eyes.

10 Forming Equations

Sometimes a problem leads to an equation which we can then solve.

EXAMPLE 1

In the diagram below form an equation, solve it and find the size of each angle.



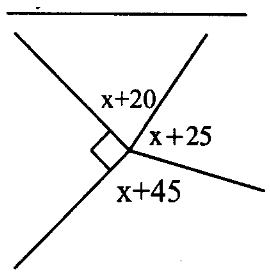
We see here that the angles form a half circle, which is 180°.

We can therefore say: $4x + 5x + x = 180$

So $10x = 180$
and $x = 18$.

And if $x = 18$, $4x = 72$ and $5x = 90$. So the angles must be 72°, 90°, 18°. We may check that these angles do actually add up to 180°.

EXAMPLE 2

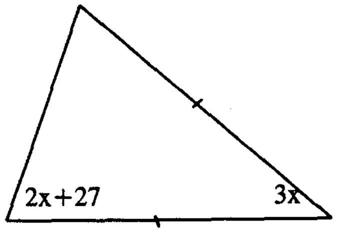


Here the four angles total 360°, and one of them is a right angle.

So $90 + x+20 + x+25 + x+45 = 360$
Then we add the x terms and the numbers up:
$3x + 180 = 360$.
So $\underline{x = 60}$.
<u>The angles are therefore 90°, 80°, 85° and 105°.</u>

EXAMPLE 3

Find the angles:



Since the triangle is isosceles the third angle must be 2x+27 also.

So $2x+27 + 2x+27 + 3x = 180$

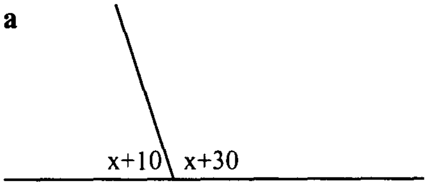
And $7x + 54 = 180$

So $x = 18$ and the angles are 54°, 63°, 63°.

EXERCISE 1

Find x and the size of the angles:

a



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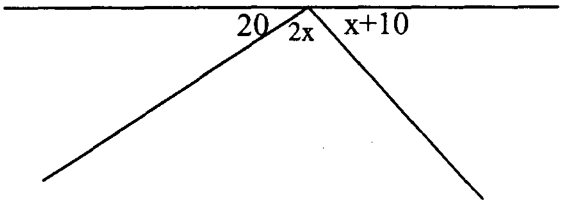

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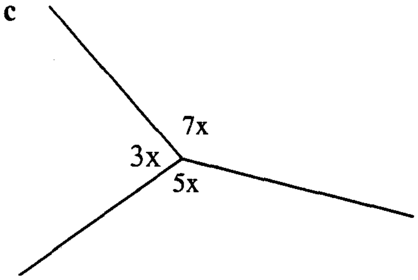

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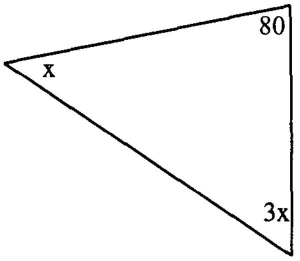

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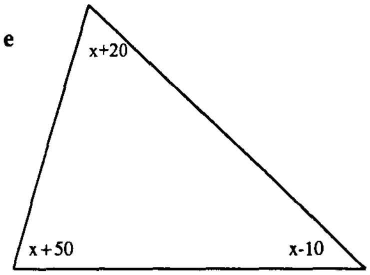

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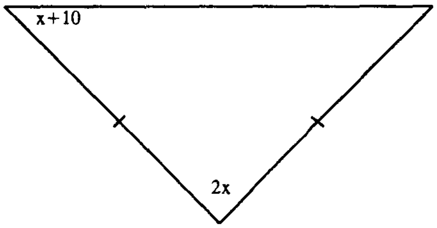

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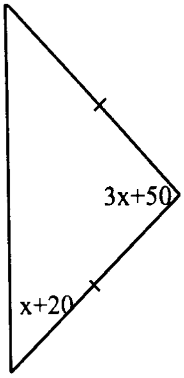
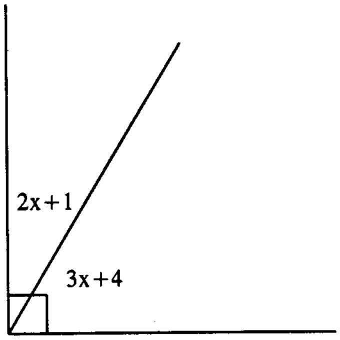
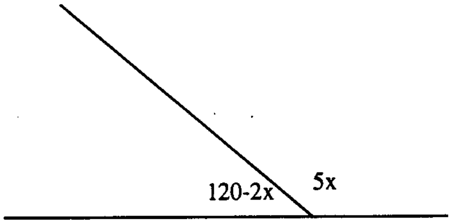


h

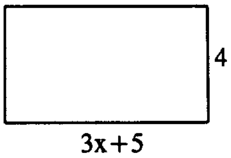


i



EXAMPLE 4

A rectangle has a base of $3x+5$ and a height of 4 units.
Find x given that **a** its perimeter is 48, **b** its area is 104.



a If the perimeter is 48: $3x+5 + 3x+5 + 4 + 4 = 48$

$6x + 18 = 48,$

$\underline{x = 5}.$

b Since the area of a rectangle is base×height, $(3x+5)\times 4 = 104$.

Dividing both sides by 4: $3x + 5 = 26,$

$\underline{x = 7}.$

EXERCISE 2

Find x in each of the following:

a An isosceles triangle has two sides of length x and one of length $x+3$ and its perimeter is 21.

b A triangle has sides x, $x+2$ and $x+4$ and its perimeter is 24.

c A right-angled triangle has a base of $2x-1$, a height of 8 and its area is 60.

d A triangle has a base of 10, height x and its area is 30.

e An isosceles triangle has sides x, x, $x+3$ and its perimeter is 63.

f A rectangle has a base of 3, a height of $2x+1$ and its area is 63.

g A number, x, when doubled and increased by 5 gives 39. Write down an equation for x and solve it.

h A number, x, is trebled and then reduced by 17. If this gives a result of 73 write down an equation for x and solve it.

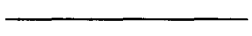
i A number, x, is reduced by 3 and then doubled to give 24. Write down an equation for x and solve it.

11 2- and 3- Dimensional Shapes

DIMENSIONS

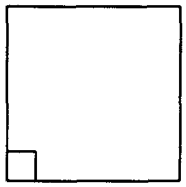
A straight line is said to be 1-dimensional because it has only one direction, or two directions if you think of it having two opposite directions.
A pencil is rather like a 1-dimensional object because it is like a straight line.

- Name some other objects which look 1-dimensional.



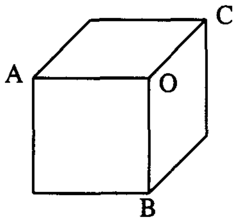
A flat area, like a table top or wall, is called a plane.
It is said to be 2-dimensional because 2 perpendicular straight lines can be placed in it. Triangles and rectangles are 2-dimensional shapes.

- Name some other objects which look 2-dimensional.



Solid or hollow objects like a cube or sphere (ball shape) are examples of 3-dimensional forms.
They can contain 3 perpendicular straight lines, like OA, OB, OC opposite.

- Name some other objects which look 3-dimensional.



Sometimes it is not obvious what dimension applies to a form:
There are forms which are both 1 and 2-dimensional, like a wavy line drawn on paper,
or 1 and 3-dimensional, like a helix (shaped like a spiral staircase),
or 2 and 3-dimensional, like the surface of a sphere.

2-DIMENSIONAL SHAPES

First let us revise our work on 2-dimensional shapes and area. You will need a sheet of A4 graph paper.

1 Using the bold lines on the graph paper draw a rectangle 18cm by 28cm.
Along the bottom edge, which should be the 28cm edge, number the centimetre marks from 0 (in the corner) to 28 (you may number every two cm if you wish).
Number from 0 to 18 up the left-hand side (in 2's if you wish).

You will remember that a point can be described by two coordinates, like (4,18).
Put a small cross at the point (4,18). That is, starting at the bottom left-hand corner (the origin) go 4 units to the right and then 18 units up.

Next you are going to plot a series of points, but as each one is drawn join it to the last point plotted. (4,18) is the first point. (Do not attempt to plot all the points first, and then join them up)

EXERCISE 1

Join to the previous point:

a (4,8)	**b** (0,8)	**c** (4,0)	**d** (4,4)	**e** (2,4)	**f** (9,18)	**g** (18,18)
h (9,0)	**i** (4,4)	**j** (24,4)	**k** (24,18)	**l** (14,10)	**m** (24,4)	**n** (28,0)
o (15,0)	**p** (15,4)	**q** (11,4)	**r** (11,0)			

You should now find that your sheet in divided into various shapes: triangles and quadrilaterals.

There is a square,
a rectangle,
2 trapezia,
a parallelogram,
5 right-angled triangles,
an isosceles triangle, and
4 other triangles.

Write the correct name inside each shape.

You may recall that:

the area of a square, rectangle or parallelogram is **base × perpendicular height**,
and that the area of a triangle is ½ **base × perpendicular height**.

EXERCISE 2

Find the area of:

a the square
b the rectangle
c the right-angled triangles
d the trapezia
e the isosceles triangle
f the other triangles

and write the answer inside the shape. You will need to split the trapezia up into a rectangle and a triangle, and for the last 5 triangles you may need to turn your sheet around.

g Now find the area of the large rectangle that contains all the shapes and use this number to find the area of the parallelogram using *By Addition and By Subtraction.*

h Check that your answer is about right by measuring the base and perpendicular height to the nearest centimetre and multiplying them.

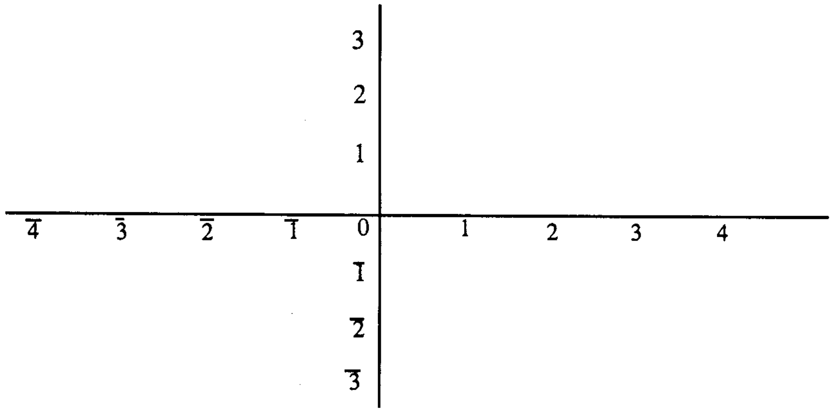
3-DIMENSIONAL SHAPES

● On another A4 graph sheet, and with the long side at the bottom, draw a horizontal line across the middle of the page so that there are 9cm above it and 9cm below it.

Next draw a vertical line through the middle of the page so that there are 14cm on each side.

Put at the intersection of the lines- this point is the Origin.

As before each centimetre is counted as 1 so number to the right of O from 1 to 14.
Just like the **number line** we have seen before we can now number to the left of O from $\bar{1}$ to $\overline{14}$.
Now number upwards from O from 1 to 9, and then downwards from O from $\bar{1}$ to $\bar{9}$.



You should have an extended version of the diagram shown above. This allows us to extend our graphs to include bar numbers.

As we have seen to plot the point D(1,7) for example we go 1 unit to the right of O and 7 units up and put a cross there with a D beside it.

EXERCISE 3

a Plot the point D as described and also the points E(2,8) and F(2,9).
Join them up to make a triangle.

Now we plot the point A($\bar{2}$,7), which means 2 units to the left of O and 7 units up.
Plot also the points B($\bar{1}$,8) and C($\bar{1}$,9) and join them to form another triangle.

Now join A to D, C to F and B to E.
This should look like a 3-dimensional shape: it is called a **triangular prism**.
Write this to the left of the figure.

b Next we will plot a **hexagonal prism**.
Plot the point A(8,5) and, joining each point to the previous point plot B(9,5), C(10,6), D(9,8), E(8,8), F(7,7) and join F to A.

Plot G(11,9), H(12,9), I(13,7), J(12,6) joining G to H, H to I and I to J.

Now join BJ, CI, DH and EG.
Write the name on the left.

A prism is a 3-dimensional object which is the same shape all the way through- the first shape was a triangle all the way through, and the second shape is a hexagon all the way through.

c For the point A with coordinates $(\bar{8},\bar{9})$ we go 8 units to the left of O and 9 units down.

Plot A and the points $B(\bar{6},\bar{9})$, $C(\bar{6},\bar{7})$, $D(\bar{8},\bar{7})$. Join these to make a square.
Plot $E(\bar{7},\bar{6})$, $F(\bar{5},\bar{6})$, $G(\bar{5},\bar{8})$ and join EF and FG.
Join DE, CF and BG to complete the shape.
Write **cube** on the left.

The point A together with the points H(4, 9), I(4, 5), J(8, 5), K(6, 3), L.(2, 3), M(2, 7) form another cube identical to the first but twice the size. Draw the larger cube in.

d The points for the next shape use all four sections of the graph. Remember, the first coordinate is found on the horizontal axis (horizontal number line) and the second coordinate tells you how far to go up or down.

Plot $A(\bar{2},\bar{2})$, $B(1,\bar{2})$, $C(1,1)$, $D(\bar{2},1)$. Join these to form a square.

Plot E(4,5), F(7,5), G(7,2) and join EF and FG. Join DE, CF and BG.
Write the name cuboid below the shape.
A cuboid is like a cube but the sides are not all the same length.

We can also draw in the hidden edges if we like: plot H(4,2) and join HA, HE, HG with a dashed line. Hidden lines are usually drawn dashed.

e Plot O(9,3), A(8,1), $B(9\frac{1}{2},1)$, C(10,2).
Join O to A, B and C.
Join AB, BC.

This is a triangular pyramid or tetrahedron.

There is a hidden line, AC. Draw a dashed line from A to C.

O is a vertex (corner) of the pyramid. A, B and C are also vertices.

Another pyramid twice the size of OABC is to be drawn with the same vertex, O.
The other vertices are at $(7,\bar{1})$, $(10,\bar{1})$ and $(11,1)$.
Draw in the second pyramid, including the hidden line.

A third pyramid three times the size of the first also has vertex O and vertices at $(6,\bar{3})$, $(10\frac{1}{2},\bar{3})$ and $(12,0)$. Draw it in.

Draw in a fourth pyramid, which is four times the size of the first, and write down the coordinates of the 3 vertices of the base (write on your graph paper).
Write "tetrahedron or triangular pyramid" below your final diagram.

f The vertices of a **rectangular-based pyramid** are $O(\overline{9\tfrac{1}{2}},6)$, $A(\overline{10},4)$, $B(\bar{9},4)$, $C(\bar{8},5)$.
Plot these points.
Note that $\overline{9\tfrac{1}{2}}$ is $\tfrac{1}{2}$ to the **right** of $\overline{10}$.

Join O to A, B and C, and join AB and BC.

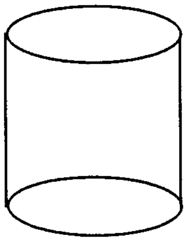
There is a hidden vertex at $D(\bar{9},5)$. Plot this point and draw in the hidden lines DA. DC and OD.

A pyramid twice the size of the first has vertices at O and $(\overline{10\tfrac{1}{2}},2)$, $(\overline{8\tfrac{1}{2}},2)$, $(\overline{6\tfrac{1}{2}},4)$, $(\overline{8\tfrac{1}{2}},4)$.
Draw this pyramid with the hidden lines also shown.

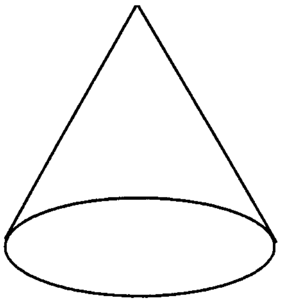
Draw in a third pyramid three times the size of the first and with vertex O (extend the lines you have already and use the squares on the graph paper to help you).
Write down the coordinates of the 4 corners of the base.

Finally repeat for a fourth pyramid, four times the size of the first. Write "square-base pyramid" below.

Two other important 3-dimensional forms are the cylinder and the cone.



cylinder



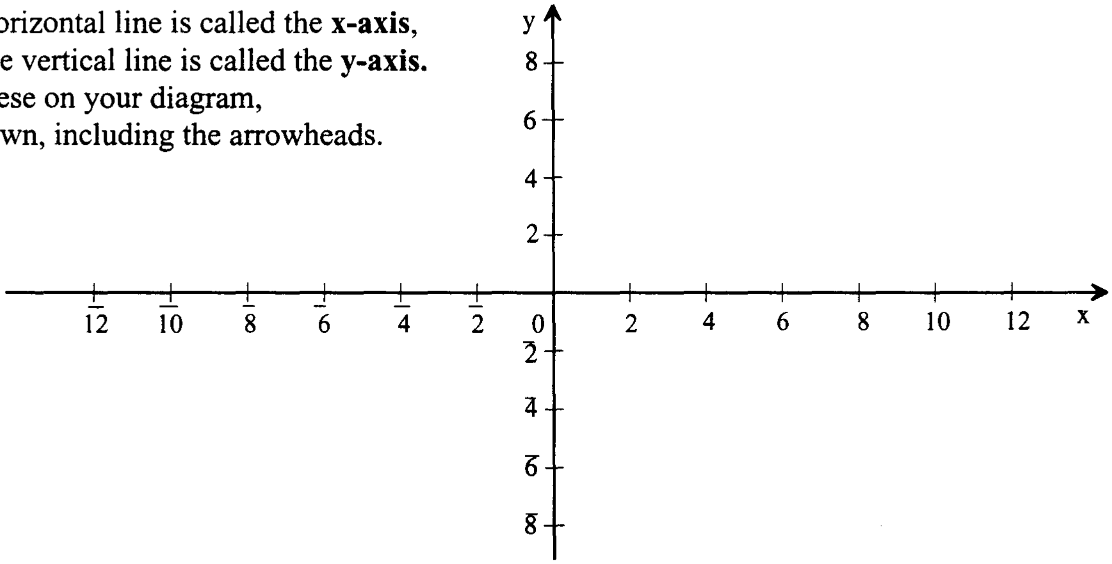
cone

12 Straight Line Graphs

The idea of converting an algebraic equation into a geometrical form came in recent times from the French philosopher and mathematician Rene Descartes in 1637.
Thus Descartes showed a remarkable way of combining Arithmetic, Algebra and Geometry.

❋ On a sheet of graph paper draw a horizontal line and a vertical line, just as you did when you drew the 3-dimensional shapes, and number from $\overline{12}$ to 12 on the horizontal line and from $\bar{8}$ to 8 on the vertical line.

The horizontal line is called the **x-axis**,
and the vertical line is called the **y-axis.**
Put these on your diagram,
as shown, including the arrowheads.



EXERCISE 1

a Suppose first of all that we want to draw the equation x=10.

For this you must find where x=10 on the x-axis and then draw a vertical line through this point. Do this and write "x=10" above it.

This is the line corresponding to x=10 since for any point on the line the first coordinate (the x coordinate) is always 10.

Mark a point on the line and write down its coordinates beside it. You should find that the x coordinate (that is the first number) is 10 whichever point you choose.

b Draw the line $\underline{x = \bar{8}}$.

Find $\bar{8}$ on the x-axis and draw a vertical line right across the page.
Write the equation above the line.

Check that it is right by writing down the coordinates of a point on the line: whatever point you choose you should find that the first coordinate is $\bar{8}$.

c Draw $\underline{y=6}$

To plot this equation find the point on the y-axis where $y=6$ and draw a horizontal line right across the page through this point. Write "y=6" on it.

Choose a point on the line and write its coordinates beside it.
The y coordinate of this point should be 6 whatever point you chose.

d Draw and label the line $y=\bar{5}$.

Find $\bar{5}$ on the y-axis and draw a horizontal line through it.
Choose a point on the line and write its coordinates beside it. Check the y-coordinate is $\bar{5}$.

So all lines with equation $x=a$ are vertical lines,
and all lines with equation $y=b$ are horizontal

The equation of the y-axis will therefore be $x=0$.

1 What do you think the equation of the x-axis will be?

SLOPING LINES

Next we consider equations which involve both x and y.

EXERCISE 2

a Draw the line $\underline{y = x}$.

Whatever value x has, y will have the same value, since $y=x$.
If $x=1$ then $y=1$. If $x=3$, $y=3$ and so on.
$x=1$, $y=1$ can be plotted on the graph: it is the point $(1,1)$.
Similarly $x=3$, $y=3$ is plotted at $(3,3)$.

Also y=x gives the points (8,8), (0,0), $(\bar{6},\bar{6})$.

Plot these 5 points on your graph paper, they should lie in a straight but sloping line. Draw a line through them and write the equation of the line at one end.

b Draw the line y = x + 3.

As before we can choose any value for x (but it should be one which is on the x-axis of your graph.
If x=1 then since y=x+3
y=1+3=4 so we get the point (1,4). Plot this point.
If x=2 y=x+3 gives
y=2+3=5 which gives the point (2,5). Plot this point.

We can show these results more neatly by making a table:

y = x + 3

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y							4	5			

The top row shows the values of x we choose.
The bottom row shows the corresponding y values.
The values we found above for x=1 and x=2 have been put in.

Copy the table into your book and fill in the remaining boxes.
Remember since y=x+3 the y value is always just 3 more than the x value, so you are just adding 3 to all the numbers in the top row.

Next we can plot each number pair as a point on the graph.
So the first point will be $(\bar{5},\bar{2})$, from the first column of your table, and so on.
This will give you 11 points on your graph, which should make a straight line.

Draw in the line and put the equation by it.

This method of making a table, choosing some x values and finding the y value that goes with each x value, can be used for drawing all equations involving x and y.

It is worth noticing the patterns in the table above: for example, the values are increasing consecutively in both rows.

We do not actually need 11 points to draw the line in. What is the fewest number of points needed to be able to draw a line?

If we know that the line is straight then two points are enough to draw the line.

c Draw the line $\underline{y = x - 2}$.

First choose a value for x, say x=8.
Then since y=x-2, y=8-2 = 6.
This gives the point (8,6) which we plot. The line goes through this point.

Next choose another x value, say x=$\bar{5}$.
Then y=$\bar{5} - 2 = \bar{7}$.
So we plot the point ($\bar{5}$, $\bar{7}$) and draw a line through the two points, extended right across the page.

Find three more points of your own for this equation: choose x, find y and plot the point, do this three times.

The points you plot should also lie on the line, confirming that the other points were correct and that y=x-2 is the equation of a straight line.

d Draw the line $\underline{y = -x}$.

If x=7 then y=-7 so we can plot (7,$\bar{7}$).
If x=$\bar{5}$ then y=-$\bar{5}$ = 5 so we plot ($\bar{5}$,5).
If x=0 then y=0 so we plot (0,0),

Check with three more points of your own, and draw in and label the line.

So the method for drawing these lines is to choose values for x, find y and plot the point. Then repeat this for other values of x.

It is best to find at least three points so that if one is wrong a correction can be made because any two points will give a line and if one point is wrong you would not know.

Do not choose the values of x too close together: the line is more accurate if the points are farther apart.

And if the y value found is not on the page you will have to choose a different x value.

Draw the following lines on the same graph page as before (do the calculations mentally):

e $y = -x + 5$ **f** $y = 8 - x$ **g** $y = -x - 3$

- Study all the sloping lines you have drawn to see what connections there are between the equations and the lines.

- Check that you agree with the following:

A) Equations with a "-x" term slope downwards to the right,

B) The other equations slope upwards to the right.

C) In each equation the term which does not contain x gives the y value where the line crosses the y-axis.

So, for example, in the equation $y = x - 2$ we have a "+x" and a "-2".
The "+x" tells us that the line slopes upwards to the right,
and the "-2" tells us that the line goes through -2 on the y-axis.

We say the intercept on the y-axis is -2.

- Check with your graph that you agree with this.

Another word for slope is **gradient.**
We say that lines sloping **upwards** to the right have a **positive gradient**,
and lines sloping **downwards** to the right have a **negative gradient**.

- Draw up some new axes now on a new page. Use the same numbering as before.

EXERCISE 3

a Plot the graph of <u>$y = -2x$</u>

As before we choose three x values.
If $x=4$ then $y=\bar{8}$,
if $x=0$ then $y=0$,
if $x=\bar{4}$ then $y=8$.

So plot the points $(4,\bar{8})$, $(0,0)$, $(\bar{4},8)$.
You should get a straight line, which you can draw in.
Plot the following lines. Do the calculations in your head:

b $y = 2x$ **c** $y = 3x$ **d** $y = -3x$ **e** $y = x$

f $y = -x$ **g** $y = \frac{1}{2}x$ **h** $y = -\frac{1}{2}x$

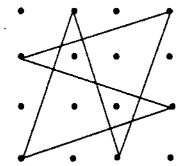
You should find that you have 8 lines on your page, all going through the origin at (0,0).

On the previous graph all the sloping lines had the same steepness- the same gradient. But here we find that the lines have different gradients.

So we see that the effect of varying the coefficient of x is to vary the gradient.

GRADIENT SQUARES

- Look carefully at the diagram below which contains lines with different slopes.



Notice that each line is a diagonal which goes 3 units horizontally and 1 unit vertically or else 3 units vertically and I horizontally.

<u>EXERCISE 4</u>

a On square spotty paper copy this diagram starting at the top right-hand corner and joining each line to the previous line.

b On the same square draw a similar pattern with the same slopes but starting at the top left corner.
c Your next gradient square will be on a 3 by 3 square of 9 dots. But this time the diagonals will be 2 units by 1 unit. Starting at the top right you should be able to draw eight continuous lines with these diagonals, ending up at the point at which you started.

d The next gradient square is on a 5 by 5 square of dots. Start again at the top right and use diagonals which are 4 units by 1 unit. You should end up where you started after 12 lines.

e On a 4 by 4 square of dots and starting at the top right draw as many joined lines as you can which are 2 units by 1 unit (you are doing well if you get 18 or more).

GRADIENTS: BY THE COMPLETION OF THE TRIANGLE

● Mark the point (4,4) on your graph. This point is on the line $y = x$.
Draw a horizontal line to the right from here. Make it 1 cm long.
Then from the right end of that line draw a vertical line upwards until it meets the line $y = x$.

You now have a small triangle with a base of 1.

● What is the height of this triangle?

The height should be 1cm, and so we say that the gradient of the line $y=x$ is 1.
❋ Do you think we would get the same answer if we had started at a different point on the line, say (8,8) or $(\bar{6},\bar{6})$?

In fact the gradient will be the same wherever you draw your triangle.

❋ Now do the same thing with the line $y = 2x$. Mark the point (2,4),
draw a line 1cm to the right,
draw a line upwards to complete the triangle.

Since the height of this triangle is 2, the gradient of the line $y=2x$ is 2.

❋ Now take the line $y = -\frac{1}{2}x$. Mark a point on it, say $(4,\bar{2})$,
draw a line 1cm to the right,
draw a line downwards to complete the triangle.

The height of this triangle is $-\frac{1}{2}$ because we have to go **downwards** $\frac{1}{2}$cm to complete the triangle. So the gradient of $y=-\frac{1}{2}x$ is $-\frac{1}{2}$.

You may have noticed by now that the gradient of the line is just the coefficient of x in each equation.

● Check with the other lines on your page that this is true, by drawing a suitable triangle for each line.

GRADIENT AND INTERCEPT

Now consider the equation $y = 2x - 10$.

In this equation we know that the gradient is 2 and that the intercept on the y-axis is -10. This information is all we need to draw the line.

EXERCISE 5

Take a new graph page but this time have the shorter edge at the bottom. The axes will go in the same place as before, but now the x-axis will be numbered from $\overline{8}$ to 8, and the y-axis from $\overline{12}$ up to 12.

a Mark the point $\overline{10}$ on the y-axis: this is the intercept of $y=2x-10$ on the y-axis.

Now we know the line has a gradient of 2 so draw a line 1 cm to the right and 2cm up and complete the triangle. The sloping line shows the direction of the line you are looking for. Extend this line across the page. Try to ensure that for each 1 cm the line goes to the right it goes 2cm up.

Your line should go through 5 on the x-axis.

b For the equation $y = -3x + g$ we start at 8 on the y-axis, go lcm to the right and 3cm down (down because the gradient is minus) and complete the triangle. Then draw in the line. Try to draw it so that for every lcm the line goes to the right it goes down 3cm.

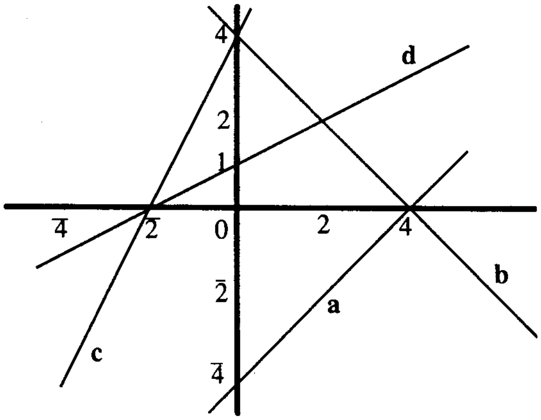
If you want to draw in a lot of triangles for each line that is fine. Or you may prefer not to draw any triangles at all, but just to use the lines on the page.

Draw the following lines by completing the triangles. Use the same graph page as above:

c $y = -2x - 10$
d $y = x + 8$
e $y = -x - 8$

f $y = \frac{1}{2}x + 4$
g $y = 3x + 8$
h $y = -4x$

i Write down the equation of the lines a, b, c, d shown below:



ALTERNATIVE METHOD USING SUBSTITUTION

We used substitution (the Sutra for this is *Specific and General)* earlier in this chapter for drawing graphs.

If we want to draw the graph of $\underline{y = 2x - 3}$ we choose a value for x, say x=2, and find the corresponding y-value.

If x=2 then y = 2X2 - 3 = 1, and this gives the point (2,1) which we can plot.

Similarly if x=3 then y=3,
if x=4 then y=5,
if x=100 then y=197,
if x=0 then y=-3,
if x=2 then y=7.

Check these results yourself.

(Note that even if x is equal to a fraction or a decimal number we can find still y).

EXERCISE 6

Draw up a new graph with axes the same as the last one.

a Plot the points (2,1), (3,3), (4,5), $(0,\bar{3})$, $(\bar{2},\bar{7})$ found above and draw the line through them.

Check that the previous method of completing the triangle gives the same graph.

As before, plotting two points is enough to draw in the line, with one extra as a check.

Draw the following lines using the method of substitution:

b $y = 2x + 5$ **c** $y = \frac{1}{2}x + 9$ **d** $y = -x - 3$ **e** $y = 1 - x$ **f** $y = 3x + 9$

If an equation does not have y as the subject it can easily be rearranged to make y the subject. For example $2x + y = 3$ can be rearranged to $y = -2x + 3$. Rearrange the next two equations to make y the subject before substituting and drawing the line.

g $y + \frac{1}{2}x = 1$ **h** $y + 2x + 11 = 0$

i Write down the coordinates of the point where

i $y=2x+5$ meets $y=3x+9$

ii $y=-x-3$ meets $y+2x+11=0$

iii $y+\frac{1}{2}x=1$ meets $y=1-x$

iv $y=2x+5$ meets $y+\frac{1}{2}x=1$.

If a point lies on a line then if the values of x and y are substituted into the equation of the line the two sides will be equal.
And if, after substituting, the two sides are not equal then the point is not on the line.

For example the point (4,5) is on the line $y = 2x - 3$ because $5 = 2X4 - 3$. Similarly the point (-2,3) is not on the line $y = 7 - 2x$ because $3 = 7 - -4$.

EXERCISE 7

Write down those points out of the four points given which are on the given line:

a $y = 3x + 5$ (4,17) (1,6) (-3,-4) (0,5)

b $2x + 3y = 8$ (1,2) (4,0) (0,1) (-2,4)

13 Charts

In this chapter we will look at how information can be obtained, how it can be shown on a graph or chart and how it can be abbreviated.

FREQUENCY TABLES

Susan has done a survey in her class to find out about shoe sizes. First she drew up a table like this:

shoe size	tally	frequency
2		
3		
4		
5		
6		

Then she asked every one in her class what their shoe size was and filled in the table:

shoe size	tally	frequency
2	||	2
3	||||	4
4	𝍸 |	6
5	𝍸 𝍸	10
6	𝍸 |||	8
		TOTAL=30

Each time she asked a member of the class their shoe size she drew a vertical line in the tally column. When there are 4 marks and another one is to be added this is done by crossing out the

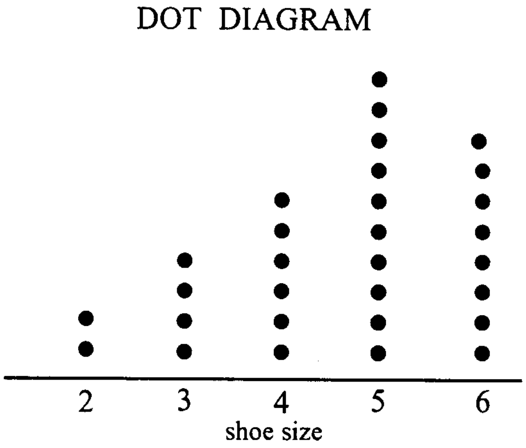
4 lines as you can see above. This makes it easy to count the total at the end because they make sets of 5.

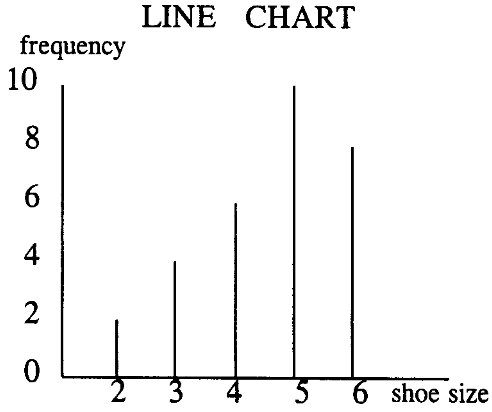
Susan put the totals for each shoe size in the third column when she had finished the survey. These numbers are called frequencies.

Finally she added up the frequencies and got 30. She then checked that there were 30 children in the class so that she had not missed anyone.

LINE CHARTS AND DOT DIAGRAMS

If information like that in the frequency table is displayed in a chart or diagram the information can be understood more easily *by mere observation.*
One way of doing this is with a line chart or a dot diagram:





Look carefully at these two charts and see how the information is represented.

The Dot Diagram has one axis only and dots are placed in the appropriate place for each person in the class.

The Line Graph has two axes and the vertical axis is always the frequency axis. Lines are drawn whose length shows the frequency.

The axes are always clearly labelled in these charts and it is usual to give a clear, brief title.

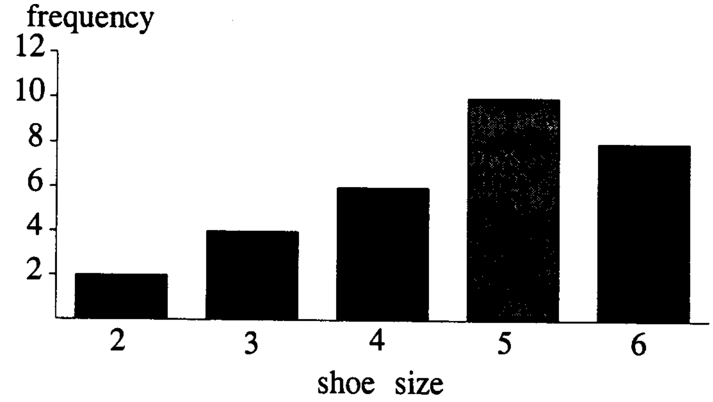
1 Carry out a survey of your own.

You can choose: shoe sizes,
family sizes,
favourite fruits,
colours of cars,
the number of people in cars going past,
measuring handspans (to the nearest centimetre or half centimetre),
number of letters in surnames
or something else which your teacher says is appropriate.

Record your results in a Frequency Table.
Draw a Dot Diagram and also a Line Graph to show your results. Keep your results safe, you may need them again later.

BAR CHARTS

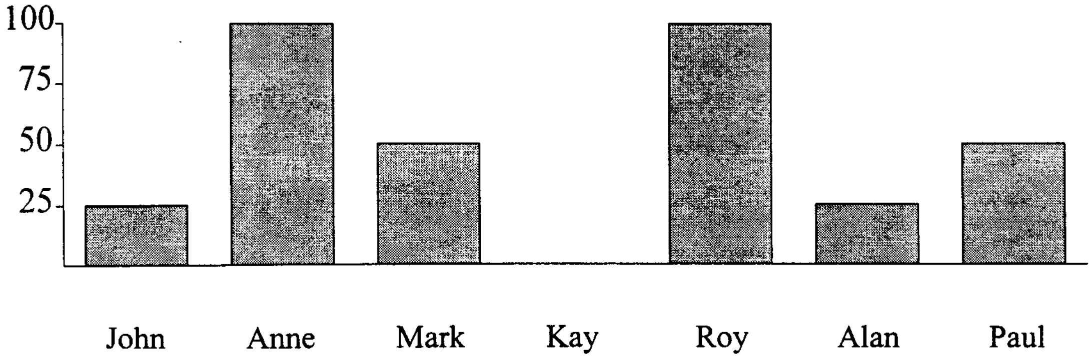
These are also easy to draw. It is like a line graph but we draw a rectangular bar instead of a line, for the frequency.
Below is a bar chart for the shoe size survey from the beginning of this chapter.



Note that the bars are of equal width.
Sometimes there is no gap between the bars.

- Use the information which you collected for your survey to draw a bar chart.
  Use a ruler, label the axes and put a title at the top. Use different colours for the bars.

Swimming Distances (in metres)



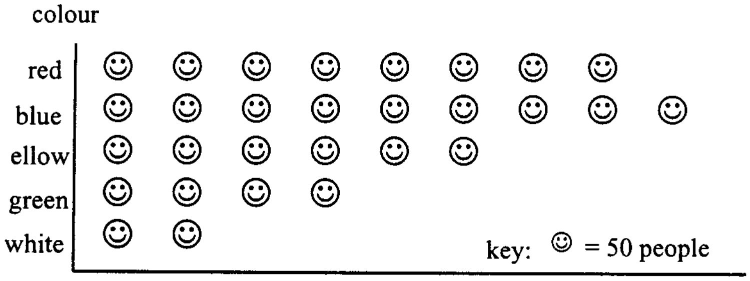
The bar chart above shows the distances a group of small children can swim in metres.

2 Who can swim the farthest?
3 How many can swim 50m?
4 Who cannot swim?

PICTOGRAMS

This is yet another way of displaying data. The diagram below, a pictogram, shows colour preferences for a number of people.

As you can see from the key each symbol represents SO people so, for example, 100 people liked white best.
You will also see a half symbol on the line for blue. This means 25 people so 475 people liked blue.



Note that this graph is different to the others because the frequency is now along the horizontal axis rather than the vertical axis.

Sometimes matchstick men or some other symbol is used instead of faces. You can use any symbol you like.

5 Represent the information from your survey in a pictogram.

AVERAGES : AND SPREAD

Information (data) like you obtained from your survey is frequently available for all sorts of subjects. There are huge amounts of data available and many ways of analysing it.

Apart from drawing graphs or charts of data we can make calculations with the numbers themselves. We can get an average (or typical) value and we can get an idea how much the data is spread out around that average value.

There are three important types of average: the mode, the median and the mean.

The mode of a collection of values is the one that occurs most often.
For example for the shoe size data we had more shoes of size 5 than any other size, so we say the mode is 5.

Sometimes there is more than one mode, or there may not be a mode at all.

EXAMPLE I

Find the mode of **4 5 5 5 7 7 8 9 13**

There are more 5's than any other number so the modal value is 5.

The median is the middle value when they are placed in order.
In Example 1 the 9 numbers are already in order and the middle value is 7: the median is 7.

In the shoe size survey there were 30 people. The Dot Diagram shown earlier shows the 30 people in order. But since there are an even number of people there will not be a middle person- there will be two people in the middle.
You will find that in this example both the middle people are in the size 5 group, so we say the median is 5.

If the two middle values were different we would find the number which is in the middle between them (which is the mean of the two numbers).

The **mean** of a set of numbers is found by adding them all up and dividing the result by the number of numbers.
In Example 1 there are 9 numbers and they add up to 63. So we find $\frac{63}{9}$ which is 7.
So the mean is 7.

If the numbers in Example 1 were numbers of apples then finding the mean is like sharing out the apples equally between 9 people. There are 63 apples so everyone would get 7 apples. The mean is the number we get if we share the numbers out equally.

Because an average is a specific value representing a collection of values this comes under the *Specific and General* formula.

The spread of a set of values is represented by the range which is the largest value minus the smallest value.

In Example 1 the range is 13-4 = 9. The bigger this number is, the more spread out the values are. If it is small then the values are packed closely together.
The range for the shoe sizes is 6-2 = 4 as the largest shoe size is 6 and the smallest is 2. The range is found using *Only the Last Terms*- the last term minus the first term.

Here is a summary of the three averages and the spread.

The **mode** is the value that occurs most often.
The **median** is the middle value when they are in order.
The **mean** is found by adding the values and dividing by the number of numbers.

The **range** is the largest value minus the smallest value.

EXAMPLE 2

Find the mode, median and mean and range for:

11 8 5 11 13 12 10 8 9 11

The mode is 11 as there are more of these than any other number.

For the median we put the numbers into order first:
5 8 8 9 10 11 11 11 12 13. Then we see that there are two middle values, 10 and 11. So the median value is the mean of 10 and 11. This is $\frac{10+11}{2}$ which is 10.5, so the median value is 10.5.

For the mean, we add the numbers to get a total of 98. Since there are 10 numbers we divide this by 10: $\frac{98}{10} = 9.8$. So the mean is 9.8.

The largest value is 13 and the smallest is 5, so the range is 8.

EXERCISE 1

Find the mode, median, mean and range for the following sets of data:

a 1 2 3 4 5 13 8 10 2 6 5 3 3

b 35 37 30 41 20 35

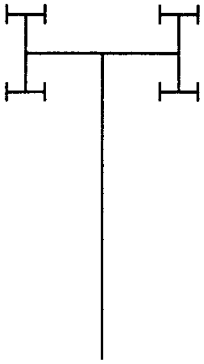
c 10.9 10.5 10.3 10.9 10.8

d 18 6 104 2 93 77

FRACTALS

We often find in nature that large structures are repeated on a smaller scale. A tree for example may divide its trunk into two parts. These may again each divide into two and so on down to the smallest twigs.

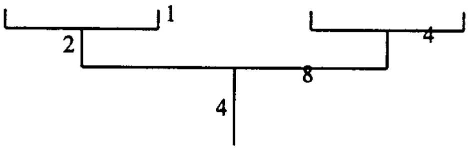
This could be modelled as follows:



We start with a vertical "trunk" which divides into two. Then each part divides into two and so on. The diagram stops after 4 divisions but these may go on indefinitely.
This is a geometrical application of the *Proportionately* Sutra together with *By One* More *than the One Before.*

A figure in which a design is repeated on a smaller and smaller scale is called a fractal. You will have met such things before in the chapter on Spirals.

- Take a sheet of graph paper and with the longer edge at the bottom mark a point near the bottom but in the middle. This point is the bottom point of the diagram shown below. Copy the symmetrical diagram, all numbers represent centimetres.



Notice that all horizontal distances are halved when they are drawn in the next level up and all vertical distances are similarly halved.

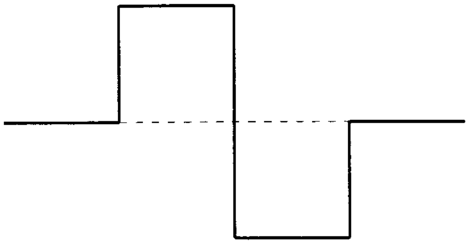
- Continue the drawing to include two more layers, each layer being half the size of the layer below.

Coastlines also show this fractal property. A coastline appears erratic from a great distance but when looked at closer although less may be in view more detail is visible so the appearance is still erratic.

We can model this as follows:

- Draw a horizontal line, in pencil, 16cm long above your previous drawing- you will need 5cm above and below this line.

- Divide this line into four equal parts and draw, in pencil, a square above the 2nd part and a square below the 3rd part as shown below.



- Then rub out the middle two quarters of the line as shown by the dashed line above.

We now have 8 lines each a quarter of the original length.
The procedure applied to the original line is now applied to each of these 8 shorter lines. ● Divide the line on the left into quarters, draw the square above the 2nd quarter and below the 3rd quarter and rub out the middle two quarters of the line.

- Do this also for the other 7 lines.

- If possible repeat the whole procedure on each of the smaller lines.

- When you have finished go over your diagram in a colour.

- Take a sheet of triangular spotty paper and draw an equilateral triangle, in pencil, with sides 18cm.

- Divide each side into 3 equal parts and draw an equilateral triangle standing on the middle part of each side and pointing outwards (you may not be able to draw one of the triangles completely). Rub out the middle third of each side.

You should now have a star with 12 sides.

- Divide each of these 12 sides similarly into 3 equal parts, construct an equilateral triangle in the middle of each and rub out the middle third of the line.

- Repeat this one more time with the resulting shape, You will need to use ruler and compasses for this.

- When you have finished go over the shape (called the Koch Snowflake) in a colour.

14 Further Multiplication

We are familiar with the *Vertically and Cross-wise* method for multiplying 2-figure numbers together and in this chapter we will see how to extend this pattern so that we can multiply numbers of any size in one line.
We will also meet some of the special numbers which are extremely useful in many areas of mathematics.

First let us revise multiplication of 2-figure numbers.

EXERCISE 1

Multiply the following:

$$\textbf{a}\ \begin{array}{r} 3\ \ 1 \\ \underline{5\ \ 1} \\ \underline{\quad\quad} \end{array} \qquad \textbf{b}\ \begin{array}{r} 2\ \ 3 \\ \underline{3\ \ 2} \\ \underline{\quad\quad} \end{array} \qquad \textbf{c}\ \begin{array}{r} 5\ \ 2 \\ \underline{4\ \ 6} \\ \underline{\quad\quad} \end{array} \qquad \textbf{d}\ \begin{array}{r} 1\ \ 7 \\ \underline{6\ \ 1} \\ \underline{\quad\quad} \end{array} \qquad \textbf{e}\ \begin{array}{r} 4\ \ 4 \\ \underline{5\ \ 4} \\ \underline{\quad\quad} \end{array} \qquad \textbf{f}\ \begin{array}{r} 8\ \ 2 \\ \underline{2\ \ 8} \\ \underline{\quad\quad} \end{array}$$

MULTIPLYING 3-FIGURE NUMBERS

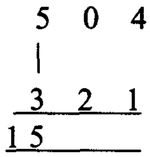
EXAMPLE 1

Find 504 × 321.

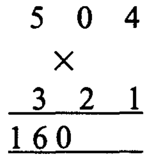
$$\begin{array}{r} 5\ \ 0\ \ 4 \\ \underline{3\ \ 2\ \ 1} \\ \underline{1\,6\,1\,7\,8\,4} \end{array}$$

The extended pattern for multiplying 3-figure numbers is as follows.

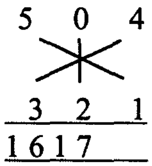
A Vertically on the left, 5×3 = **15**.



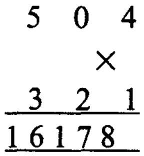
B Then cross-wise on the left,
5×2 + 0×3 = 10.
Combining the 15 and 10:
$15\underset{\cup}{,}1\,0 = \mathbf{160}$.



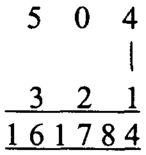
C Next we take 3 products and add them up,
5×1 + 0×2 + 4×3 = 17. And $160\underset{\cup}{,}17 = \mathbf{1617}$.
(actually we are gathering up the hundreds by multiplying hundreds by units, tens by tens and units by hundreds)



D Next we multiply cross-wise on the right,
0×1 + 4×2 = 8: 1617,8 = **16178**.

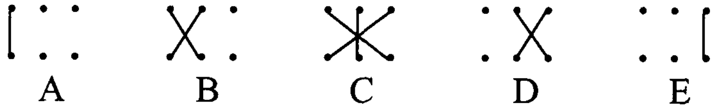


E Finally, vertically on the right,
4×1 = 4: 16178,4 = **161784**.



Note the symmetry in the 5 steps:
first there is 1 product, then 2, then 3, then 2, then 1.

We may summarise these steps as:



EXAMPLE 2

$$\begin{array}{r} 3\ 2\ 1 \\ \underline{3\ 2\ 1} \times \\ \underline{103041} \end{array}$$

The 5 results are 9,12,10,4,1.
The mental steps are 9
$9\underset{\cup}{,}12 = 102$
$102\underset{\cup}{,}10 = 1030$
1030,4,1 = 103041

EXAMPLE 3

Find 123 × 45.

This can be done with the moving multiplier method or by the smaller vertical and cross-wise pattern, treating 12 in 123 as a single digit.
Alternatively, we can put 045 for 45 and use the extended vertical and cross-wise pattern:

$$\begin{array}{rl} 1\,2\,3 & \\ \underline{0\,4\,5} & \text{For the 5 steps we get } 0,4,13,22,15. \\ \underline{5535} & \text{Mentally we think } 4;\ 53;\ 552;\ 5535. \end{array}$$

EXERCISE 2

Multiply (there are no carries in the first few sums):

a 1 2 1	**b** 1 3 1	**c** 1 2 1	**d** 3 1 3	**e** 2 1 2	**f** 1 2 3
1 3 1	2 1 2	2 2 2	1 2 1	3 1 3	3 2 1
____	____	____	____	____	____
g 2 1 2	**h** 2 2 2	**i** 2 4 6	**j** 1 0 5	**k** 1 0 6	**l** 5 1 5
4 1 4	3 3 3	3 3 3	5 0 7	2 2 2	5 5 5
____	____	____	____	____	____
m 4 4 4	**n** 3 2 1	**o** 1 2 3	**p** 1 2 4	**q** 1 3 7	**r** 1 3 1
7 7 7	3 2 1	2 7 1	3 5 6	8 0 3	7 7 1
____	____	____	____	____	____

FROM RIGHT TO LEFT

EXAMPLE 4

We can also calculate from right to left if we prefer, though for mental calculations left to right is much better.

$$\begin{array}{ccccc} & & 2 & 3 & 4 \\ \hline & & 2 & 3 & 4 \\ \hline 5 & 4 & 7 & 5 & 6 \\ \hline {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 2} & {\scriptstyle 1} & \end{array}$$

We simply do the same operations but start at the right side:
4×4 = 16, put down 6 and carry 1 to the left.
3×4 + 4×3 = 24, 24 + carried 1 = 25, put down 5 and carry 2.
And so on.

EXERCISE 3

Multiply the following from right to left:

a 444 × 333 **b** 543 × 345 **c** 707 × 333 **d** 623 × 632

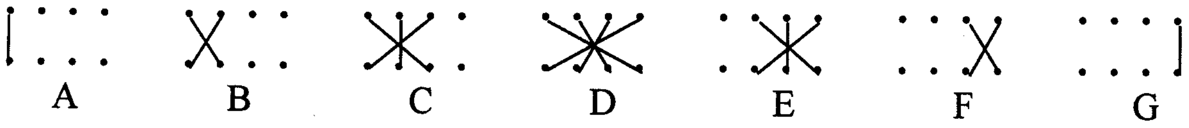
4-FIGURE NUMBERS

Once the vertical and cross-wise method is understood it can be extended to multiply numbers of any size. We here extend the pattern one stage further, and multiply two 4-figure numbers.

EXAMPLE 5

$$\begin{array}{r} 3\ 2\ 0\ 1 \\ \underline{4\ 3\ 0\ 2} \times \\ \underline{13770702} \end{array}$$

The 7 steps are illustrated as follows:



Working from left to right we get:

A. 3×4 = 12
B. 3×3 + 2×4 = 17
C. 3×0 + 2×3 + 0×4 = 6
D. 3×2 + 2×0 + 0×3 + 1×4 = 10
E. 2×2 + 0×0 + 1×3 = 7
F. 0×2 + 1×0 = 0
G. 1×2 = 2

The mental steps are therefore

$12,\underset{\cup}{1}\,7 = 137$
$137,6 = 1376$
$1376,\underset{\cup}{1}0 = 13770$
$13770,7,0,2 = \underline{13770702}$

Or calculating from right to left we get the numbers from G to A shown above:

$$\begin{array}{r} 3\quad 2\quad 0\quad 1 \\ \underline{4\quad 3\quad 0\quad 2} \times \\ \underline{1\,3\,7\;\,7\,0\;\,7\,0\;\,2} \\ 1\quad\;\; 1\qquad\qquad\quad \end{array}$$

EXERCISE 4

Multiply the following from left to right or from right to left:

a	2131	**b**	2021	**c**	3201	**d**	5113
	3022		1122		4012		5331

SQUARING

You will recall how we find the square of a 2-figure number by summing its Duplexes from left to right.

EXAMPLE 6

$37^2 = \underline{1369}$

The duplex of 3 = $3^2 = 9$,
the duplex of 37 = $2\times3\times7 = 42$,
the duplex of 7 = $7^2 = 49$.

Summing these: $9\underset{\cup}{,}4\,2 = 132$, $\quad 132\underset{\cup}{,}4\,9 = 1369$.

Practice a few of these to remind yourself of the method.

EXERCISE 5

Square:

a 62 **b** 71 **c** 26 **d** 34 **e** 56 **f** 83

We can also find the duplex of 3-figure numbers or bigger.

For 3 figures D is **twice the product of the outer pair + the square of the middle digit**,
e.g. $D(137) = 2\times1\times7 + 3^2 = \mathbf{23}$;

for 4 figures D is **twice the product of the outer pair + twice the product of the inner pair**,
e.g. $D(1034) = 2\times1\times4 + 2\times0\times3 = \mathbf{8}$;

$D(10345) = 2\times1\times5 + 2\times0\times4 + 3^2 = \mathbf{19}$;

and so on.

EXERCISE 6

Find the duplex of the following numbers:

a 5
b 23
c 55
d 2
e 14
f 234
g 282
h 77
i 304
j 270
k 1234
l 3032
m 7130
n 20121
o 32104

As with 2-figure numbers the square of a number is just the total of its duplexes.

EXAMPLE 7

$341^2 = \underline{116281}$

Here we have a 3-figure number:

$D(3) = 9$, $D(34) = 24$, $D(341) = 22$, $D(41) = 8$, $D(1) = 1$.

Mentally:

$$9\underset{\cup}{,}24 = 114$$
$$114\underset{\cup}{,}22 = 1162$$
$$1162,8,1 = 116281$$

EXAMPLE 8

$4332^2 = \underline{18766224}$

D(4) = 16, D(43) = 24, D(433) = 33, D(4332) = 34,
D(332) = 21, D(32) = 12, D(2) = 4.

Mentally

$16\underset{\cup}{,}24 = 184$

$184\underset{\cup}{,}33 = 1873$

$1873\underset{\cup}{,}34 = 18764$

$18764\underset{\cup}{,}21 = 187661$

$187661\underset{\cup}{,}12 = 1876622$

$1876622,4 = 18766224$

EXERCISE 7

Square the following numbers:

a 212 **b** 131 **c** 204 **d** 513 **e** 263 **f** 264 **g** 313 **h** 217

i 3103 **j** 2132 **k** 1414 **l** 4144 **m** Find x given that $\mathbf{x23^2 = 388129}$

n Find b, c and d given that $\mathbf{b15^2 = 17cccd}$

SPECIAL NUMBERS

Some multiplications are particularly easy.

EXAMPLE 9

23 × 101 = <u>2323</u>

To multiply 23 by 101 we need 23 hundreds and 23 ones, which gives 2323.

The effect of multiplying any 2-figure by 101 is simply to make it repeat itself.

EXAMPLE 10

Similarly $69 \times 101 = \underline{6969}$

EXAMPLE 11

And $473 \times 1001 = \underline{473473}$

Here we have a 3-figure number multiplied by 1001 which makes the 3-figure number repeat itself.

EXAMPLE 12

$47 \times 1001 = \underline{47047}$

Here, because we want to multiply by 1001, we can think of 47 as 047.
So we get 047047, or just 47047.

EXAMPLE 13

$123 \times 101 = 123\underset{\cup}{,}123 = \underline{12423}$

Here we have 12300 + 123 so the 1 has to be carried over.

EXAMPLE 14

$28 \times 10101 = \underline{282828}$

EXERCISE 8

Find:

a 46 × 101 **b** 246 × 1001 **c** 321 × 1001 **d** 439 × 1001 **e** 3456 × 10001

f 53 × 10101 **g** 74 × 1001 **h** 73 × 101 **i** 29 × 1010101 **j** 277 × 101

k 521 × 101 **l** 616 × 101

PROPORTIONATELY

EXAMPLE 15

$43 \times 201 = \underline{8643}$.

Here we bring in the *Proportionately* formula: because we want to multiply by 201 rather than 101 we must put twice 43 (which is 86) then 43.

EXAMPLE 16

$31 \times 10203 = \underline{316293}$ we have 31×1, 31×2, 31×3.

EXERCISE 9

Find:

a 54×201
b 333×1003
c 41×10201
d 33×30201
e 17×20102
f 13×105
g 234×2001
h 234×1003
i 43×203

DISGUISES

Now it is possible for a sum to be of the above type without it being obvious- it may be disguised. If we know the factors of some of these special numbers (like 1001, 203 etc.) we can make some sums very easy.

Suppose for example you know that 3 x 67 = 201.

EXAMPLE 17

$93 \times 67 = \underline{6231}$.

$$\begin{aligned} \text{Since} \quad 3 \times 67 &= 201, \\ \text{therefore} \quad 93 \times 67 &= 31 \times 3 \times 67 \\ &= 31 \times 201 \\ &= 6231 \end{aligned}$$

In other words, we recognise that one of the special numbers (201 in this case) is contained in the sum (as 3 X 67).

Now suppose we know that 3 x 37 =111.

EXAMPLE 18

$24 \times 37 = \underline{888}$

We know that $3 \times 37 = 111$, which is a number very easy to multiply.

$$\begin{aligned} \text{So } 24 \times 37 &= 8 \times 3 \times 37 \\ &= 8 \times 111 \\ &= 888. \end{aligned}$$

Also $19{\times}21 = 399 = 40\bar{1}$.

EXAMPLE 19

$38 \times 63 = \underline{2394}$

Since $38 \times 63 = 2 \times 19 \times 3 \times 21 = 6 \times 19 \times 21 = 6 \times 40\bar{1} = 240\bar{6} = 2394$.

If we know the factors of these special numbers we can make good use of them when they come up in a sum, and they arise quite frequently.

Below is a list of a few of these numbers with their factors:

$67 \times 3 = 201$
$43 \times 7 = 301$
$7 \times 11 \times 13 = 1001$
$3 \times 37 = 111$

$17 \times 6 = 102$
$13 \times 8 = 104$
$29 \times 7 = 203$
$31 \times 13 = 403$

$11 \times 9 = 10\bar{1}$
$19 \times 21 = 40\bar{1}$
$23 \times 13 = 30\bar{1}$
$27 \times 37 = 100\bar{1}$

EXAMPLE 20

$62 \times 39 = \underline{2418}$

We see 31×13 contained in this sum:

$$\begin{aligned} 62 \times 39 &= 2{\times}31 \times 3{\times}13 \\ &= 2{\times}3 \times 31{\times}13 \\ &= 6 \times 403 \\ &= 2418 \end{aligned}$$

EXERCISE 10

Use the special numbers to find:

a 29 × 28
b 35 × 43
c 67 × 93
d 86 × 63
e 77 × 43
f 26 × 77
g 34 × 72
h 57 × 21
i 58 × 63
j 26 × 23
k 134 × 36
l 56 × 29
m 93 × 65
n 54 × 74
o 39 × 64
p 51 × 42

The special number 1001 explains how the trick shown at the beginning of the last chapter works. Repeating the 3-figure number is equivalent to multiplying it by 1001, and since 7 divides into 1001, 7 will divide into the 6-figure number.

15 Combining Fractions

Fractions may be combined by adding, subtracting, multiplying or dividing them.

ADDITION AND SUBTRACTION

If the denominators of two fractions are the same it is very easy to add or subtract them.

EXAMPLE 1

Find **a** $\frac{3}{7}+\frac{2}{7}$ **b** $2\frac{2}{5}+3\frac{4}{5}$.

a Clearly three sevenths and two sevenths make five sevenths, or $\frac{3}{7}+\frac{2}{7}=\underline{\frac{5}{7}}$.

b Similarly $2\frac{2}{5}+3\frac{4}{5}=5\frac{6}{5}$ we add the whole numbers and the fractions separately. And since $\frac{6}{5}=1\frac{1}{5}$ this simplifies further to $\underline{6\frac{1}{5}}$.

EXAMPLE 2

Find **a** $7\frac{8}{9}-3\frac{2}{9}$ **b** $3\frac{4}{7}-\frac{6}{7}$.

a $7\frac{8}{9}-3\frac{2}{9}=4\frac{6}{9}=\underline{4\frac{2}{3}}$ We subtract the whole numbers, and subtract the numerators of the fractions.

We can also cancel $\frac{6}{9}$ down to $\frac{2}{3}$ ☞

b $3\frac{4}{7}-\frac{6}{7}=\underline{2\frac{5}{7}}$ Here we want to take $\frac{6}{7}$ from $\frac{4}{7}$ so we take away $\frac{4}{7}$ and we still have to take $\frac{2}{7}$ away so we take it from one of the 3 whole ones. This leaves only 2 whole ones and $\frac{5}{7}$ of the whole one from which we subtracted the $\frac{2}{7}$.

➡ EXAMPLE 3

Find $4\frac{3}{4}-7\frac{1}{4}+8\frac{3}{4}$.

Where there are three or more fractions to combine we can combine the first two and then combine the answer with the third. But in this case, to avoid getting involved with minus numbers we may prefer to add up the first and third and then subtract the second:

So $4\frac{3}{4}+8\frac{3}{4}=12\frac{6}{4}$. Then $12\frac{6}{4}-7\frac{1}{4}=5\frac{5}{4}=\underline{6\frac{1}{4}}$.

EXERCISE 1

Add or subtract the following fractions, giving your answers as mixed numbers and cancelled down where possible:

a $\frac{2}{5}+\frac{2}{5}$

b $\frac{3}{8}+\frac{7}{8}$

c $\frac{5}{7}+\frac{5}{7}$

d $\frac{5}{6}+\frac{5}{6}$

e $3\frac{1}{5}+4\frac{3}{5}$

f $5\frac{2}{3}+2\frac{2}{3}$

g $4\frac{1}{3}+\frac{2}{3}$

h $\frac{3}{4}+1\frac{1}{4}+3\frac{1}{4}$

i $\frac{10}{11}-\frac{4}{11}$

j $3\frac{5}{8}-1\frac{1}{8}$

k $7\frac{7}{10}-3\frac{6}{10}$

l $5\frac{4}{9}-\frac{6}{9}$

m $3\frac{1}{3}-1\frac{2}{3}$

n $8\frac{1}{15}-3\frac{4}{15}$

o $5-2\frac{2}{3}$

p $60-\frac{1}{5}$

q $1\frac{2}{5}+2\frac{4}{5}-1\frac{3}{5}$

r $3\frac{3}{7}+\frac{2}{7}-1\frac{6}{7}$

s $6\frac{1}{3}-3\frac{2}{3}+8\frac{1}{3}$

t $4-1\frac{3}{4}+3\frac{1}{4}$

u $8-1\frac{1}{6}-2\frac{1}{6}$

v $3-5\frac{1}{2}+9$

w $1-3\frac{4}{5}+10\frac{1}{5}$

x $1\frac{2}{3}+7\frac{1}{3}-2\frac{1}{3}+\frac{2}{3}$

Where the denominators are the same all we need is *By Addition and By Subtraction* to do the addition or subtraction. Where the denominators are different however we use the *Vertically and Cross-wise formula.*

EXAMPLE 4

Find **a** $\frac{2}{3}+\frac{1}{7}$ **b** $7\frac{4}{5}+2\frac{1}{3}$.

a We multiply cross-wise and add the get the numerator: $2\times7+1\times3=17$,
then multiply the denominators to get the denominator: $3\times7=21$.

So $\frac{2}{3}+\frac{1}{7}=\underline{\frac{17}{21}}$.

The reason why this works is that in order to add the fractions we must get the denominators to be equal, and we do this by multiplying top and bottom of $\frac{2}{3}$ by 7 (to get a denominator of 21) and the top and bottom of $\frac{1}{7}$ by 3 (to get the same denominator of 21). So each numerator gets multiplied by the other denominator, which is exactly what we did.

b $7\frac{4}{5}+2\frac{1}{3}=9\frac{17}{15}=\underline{10\frac{2}{15}}$ Here we can add the whole parts and the fractions separately: for the whole ones 7+2 = 9 and for the fractions: $4\times3+1\times5=17=$ numerator,
and $5\times3=15=$ denominator.

EXAMPLE 5

Find **a** $\frac{6}{7} - \frac{1}{4}$ **b** $5\frac{4}{5} - 1\frac{3}{4}$ **c** $4\frac{1}{3} - 1\frac{2}{5}$.

a Subtraction is the same except we cross-multiply and **subtract** rather than add:

$$\frac{6}{7} - \frac{1}{4} = \frac{6\times4 - 1\times7}{7\times4} = \underline{\frac{17}{28}}.$$

b $5\frac{4}{5} - 1\frac{3}{4} = 4\frac{4\times4 - 3\times5}{5\times4} = \underline{4\frac{1}{20}}$ Similarly here but deal with the whole parts first.

c $4\frac{1}{3} - 1\frac{2}{5} = 3\frac{1\times5 - 2\times3}{3\times5} = 3\frac{\bar{1}}{15} = \underline{2\frac{14}{15}}$. Here we get a minus numerator, but it is easily dealt with by taking $\frac{1}{15}$ from one of the whole ones.

Alternatively, to avoid the minus number here, put both fractions into top-heavy form and subtract. This will mean dealing with larger numbers however.

EXERCISE 2

Combine the following, cancelling down or leaving as mixed numbers where necessary:

a $\frac{2}{5} + \frac{1}{4}$
b $\frac{3}{8} + \frac{2}{5}$
c $\frac{1}{2} + \frac{2}{5}$
d $\frac{6}{7} + \frac{1}{2}$
e $\frac{4}{5} + \frac{2}{3}$
f $1\frac{1}{3} + 2\frac{1}{4}$
g $3\frac{3}{4} + 2\frac{1}{3}$
h $\frac{5}{6} + 2\frac{1}{7}$
i $\frac{3}{4} + \frac{5}{6}$
j $5\frac{1}{2} + 2\frac{3}{8}$
k $3\frac{4}{5} + 1\frac{2}{3}$
l $1\frac{1}{5} + \frac{3}{10}$
m $\frac{3}{5} - \frac{2}{7}$
n $\frac{8}{9} - \frac{1}{2}$
o $\frac{3}{4} - \frac{1}{20}$
p $5\frac{3}{5} - 2\frac{1}{2}$
q $10\frac{2}{3} - 1\frac{2}{5}$
r $4\frac{1}{2} - 1\frac{2}{3}$
s $5\frac{1}{10} - 2\frac{1}{3}$
t $1\frac{3}{10} - \frac{2}{3}$
u $\frac{5}{12} + \frac{7}{18}$

COMPARING FRACTIONS

Some times we need to know whether one fraction is greater or smaller than another, or we may have to put fractions in order of size.

EXAMPLE 6

Put the fractions $\frac{4}{5}$, $\frac{2}{3}$, $\frac{5}{6}$ in ascending order.

Looking at the first two fractions we cross-multiply and subtract as if we wanted to subtract the fractions. If we find the subtraction is possible without going into negative numbers then the first fraction must be greater: since 4×3 is greater than 2×5, $\frac{4}{5}$ must be greater than $\frac{2}{3}$.

Doing the same thing with $\frac{2}{3}$ and $\frac{5}{6}$ we find that 2×6 is less than 5×3, so $\frac{5}{6}$ is greater than $\frac{2}{3}$.

If we now cross-multiply $\frac{4}{5}$ with $\frac{5}{6}$ we find that $\frac{5}{6}$ is greater.

So in ascending order the fractions are: $\underline{\frac{2}{3}, \frac{4}{5}, \frac{5}{6}}$.

EXERCISE 3

Put the following fractions in ascending order:

a $\frac{1}{3}, \frac{2}{5}$ **b** $\frac{3}{4}, \frac{8}{11}$ **c** $\frac{2}{3}, \frac{7}{12}, \frac{3}{4}$ **d** $\frac{5}{6}, \frac{5}{8}, \frac{6}{7}$

A SIMPLIFICATION

In the last question of Exercise 2 the numbers were rather large and some cancelling had to be done at the end. Where the denominators of two fractions are not relatively prime the working can be simplified as shown in the next example.

EXAMPLE 7

The denominators in $\frac{5}{12} + \frac{7}{18}$ are not relatively prime: there is a common factor of 6. We divide both denominators by this common factor and put these numbers below the denominators:

$$\underset{(2)}{\frac{5}{12}} + \underset{(3)}{\frac{7}{18}} = \frac{5\times3 + 7\times2}{12\times3} = \frac{29}{36}.$$

So we put 2 and 3 below 12 and 18.
Then when cross-multiplying we use the 2 and 3 rather than the 12 and 18.
For the denominator of the answer we cross-multiply in the denominators:
either 12×3 or 18×2, both give 36.

Subtraction of fractions with denominators which are not relatively prime is done in just the same way, except we subtract in the numerator as before.

EXERCISE 4

Use this simplification to add or subtract the following:

a $\frac{1}{3}+\frac{4}{9}$	**b** $\frac{3}{8}+\frac{1}{6}$	**c** $\frac{3}{5}+\frac{3}{10}$	**d** $\frac{5}{6}-\frac{3}{4}$
e $\frac{5}{6}+\frac{3}{4}$	**f** $\frac{5}{18}-\frac{1}{27}$	**g** $3\frac{3}{4}-1\frac{1}{8}$	**h** $\frac{7}{36}-\frac{11}{60}$

EXERCISE 5

Try the following problems involving fractions:

a How much less than 3 is the sum of $\frac{1}{3}$ and $\frac{1}{4}$?

b If I spend $\frac{1}{3}$ of the day sleeping and $\frac{5}{12}$ of the day working what fraction of the day is left, and how many hours is that?

c I spent $\frac{1}{3}$ of my money at the stationers and $\frac{5}{12}$ at the grocers and had £1.20 left. How much money did I have to begin with?

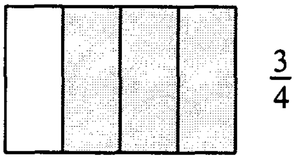
MULTIPLICATION AND DIVISION

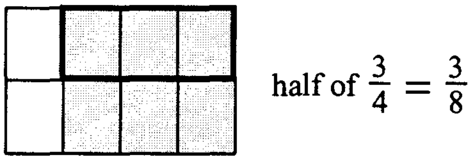
EXAMPLE 8

Find **a** $\frac{1}{2}\times\frac{3}{4}$ **b** $1\frac{1}{4}\times 2\frac{3}{5}$.

a $\frac{1}{2}\times\frac{3}{4}=\frac{1\times 3}{2\times 4}=\frac{3}{8}$ we simply multiply the numerators to get the numerator of the answer, and multiply the denominators to get the denominator of the answer.

We can verify this by showing that ½ of ¾ is $\frac{3}{8}$:





b $1\frac{1}{4}\times 2\frac{3}{5}=\frac{5}{4}\times\frac{13}{5}=\frac{65}{20}=3\frac{5}{20}=\underline{3\frac{1}{4}}$. Here we must change the mixed numbers into top-heavy fractions at the beginning, and then we can proceed as before.

EXAMPLE 9

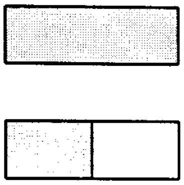
Find **a** $\frac{3}{4} \div \frac{2}{5}$ **b** $1\frac{1}{2} \div \frac{1}{4}$.

a $\frac{3}{4} \div \frac{2}{5} = \frac{3\times5}{2\times4} = \underline{\frac{15}{8}} = \underline{1\frac{7}{8}}$. We simply cross-multiply and put the first product over the second product.

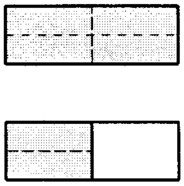
In fact either answer, $\frac{15}{8}$ or $1\frac{7}{8}$, is acceptable depending on the wording of the question.

b $1\frac{1}{2} \div \frac{1}{4} = \frac{3}{2} \div \frac{1}{4} = \frac{12}{2} = \underline{6}$. As with multiplication we first change mixed numbers into top-heavy fractions.

We can verify this by showing that when 1½ is divided into ¼'s there are 6 of them:



1½



6 quarters in 1½

Since multiplication and division are opposite processes the above methods can also be verified as follows:

Example 8 shows that $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$ so that $\frac{3}{8} \div \frac{3}{4}$ should be $\frac{1}{2}$. But $\frac{3}{8} \div \frac{3}{4} = \frac{12}{24} = \frac{1}{2}$. We have multiplied $\frac{1}{2}$ by $\frac{3}{4}$ and divided by $\frac{3}{4}$ which gives us our $\frac{1}{2}$ back.

EXERCISE 6

Multiply or divide leaving your answer **cancelled down and as top-heavy fractions** where appropriate:

a $\frac{2}{3} \times \frac{4}{5}$
b $\frac{6}{7} \times \frac{1}{11}$
c $\frac{2}{5} \times \frac{7}{8}$
d $\frac{3}{2} \times \frac{7}{9}$

e $2\frac{1}{3} \times 3\frac{1}{2}$
f $3\frac{3}{4} \times 1\frac{2}{3}$
g $4\frac{1}{4} \times \frac{2}{5}$
h $\frac{1}{6} \times 1\frac{3}{10}$

i $\frac{2}{5} \div \frac{3}{7}$
j $\frac{7}{9} \div \frac{1}{3}$
k $\frac{1}{11} \div \frac{2}{11}$
l $\frac{3}{7} \div \frac{7}{3}$

m $3\frac{1}{3} \div 3\frac{3}{4}$
n $3\frac{1}{4} \div 1\frac{1}{7}$
o $\frac{5}{6} \div 1\frac{1}{3}$
p $3\frac{1}{3} \div \frac{20}{3}$

A SIMPLIFYING DEVICE

In question **f** in the last exercise you will have had $\frac{15}{4} \times \frac{5}{3}$ and looking at the 15 and the 3 here, which are not relatively prime, we cancel by the common factor, 3, which gives $\frac{5}{4} \times \frac{5}{1} = \frac{25}{4}$.
For multiplication we can cancel any number on top with any number on the bottom.
And sometimes we can cancel both ways: e.g. $\frac{10}{21} \times \frac{9}{25}$ can be cancelled by 5 one way and by 3 the other way, giving $\frac{2}{7} \times \frac{3}{5} = \frac{6}{35}$.

Also, in question **m** we had $\frac{10}{3} \div \frac{15}{4}$. Here we can cancel the 10 and 15 to get $\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$.
For division we can cancel any two numbers which are either both on top or both on the bottom.
And we can sometimes cancel on the top and on the bottom: e.g.. $\frac{6}{35} \div \frac{9}{14}$ can be cancelled by 3 on the top and by 7 on the bottom, giving $\frac{2}{5} \div \frac{3}{2} = \frac{4}{15}$.

EXERCISE 7

Use this cancelling down to find the following:

a $\frac{6}{5} \times \frac{1}{9}$

b $\frac{7}{12} \times \frac{8}{11}$

c $2\frac{2}{3} \times \frac{1}{12}$

d $\frac{10}{21} \times \frac{14}{15}$

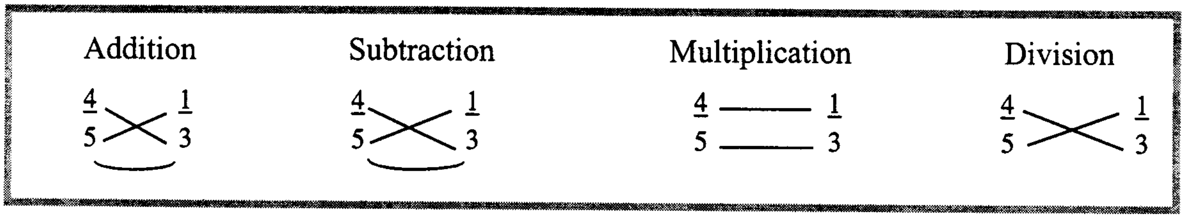
e $\frac{8}{9} \div \frac{10}{7}$

f $\frac{6}{25} \div \frac{7}{15}$

g $\frac{6}{11} \div \frac{15}{22}$

h $1\frac{3}{7} \div 4\frac{2}{7}$

We can summarise the methods of adding, subtracting, multiplying and dividing fractions as follows:



EXERCISE 8

Try the following problems involving fractions:

a Find four fifths of twelve hundred.

b I have read $\frac{2}{3}$ of a book of 444 pages. How many pages have I left to read?

c The length of a room is 2½ times the height and the width is $\frac{3}{5}$ of the length. If the height of the room is 2.8m find the perimeter of the floor.

d Three sisters shared a cake. The eldest had $\frac{1}{3}$ of it and the next sister had $\frac{3}{4}$ of the remainder. What fraction of the cake was left for the third sister?

16 Arithmetical Operations

Arithmetical Operations refers to the various ways in which numbers can be combined together.

Addition, subtraction, multiplication, division and squaring are all examples of arithmetical operations.

THE ORDER OF OPERATIONS

When we add or multiply the order is not important: 3 + 4 is the same as 4 + 3
and 3 X 4 is the same as 4 X 3.

For subtraction and division the order is important: 4 - 2 is different from 2 - 4
and 4 = 2 is different from 2 = 4.

Similarly when adding a series of numbers we can add in any order we like:
in adding 52 + 39 + 8 we may prefer to add the 52 and 8 first and then the 39.

And in multiplying 2 x 18 x 5 we may prefer to multiply 2 by 5 first and then multiply by 18.
In this section we are going to look at how to work out sums which contain a mixture of different operations.

- What is the value of 2 x 3+4 and 2+3 x 4?

In fact there is a convention (rule) that if there is a mixture of addition and multiplication then the multiplication is always done first.

So the answers to the sums above are 10 and 14 because in the second sum we first multiply 3 by 4 and then add on the 2 afterwards.

EXAMPLE 1

Find 3 × 4 + 4 × 5.

We see two multiplications here: 3×4 and 4×5 and they must both be done before adding.

Since 3×4=12 and 4×5=20 the sum becomes 12 + 20 which gives the answer 32.

EXERCISE 1

Find the value of:

a $7 + 3 \times 5$ **b** $8 + 4 \times 6 + 3$ **c** $5 \times 9 + 3 \times 3$ **d** $19 + 88 \times 96$

e $9\times7+4\times7+7\times3$ **f** $4 + 3 \times 3 \times 7$ **g** $98 + 98 \times 98$ **h** $102\times103+104$

The convention referred to above is actually wider than stated:

All multiplications and divisions should be done before additions and subtractions.

EXAMPLE 2

Evaluate $50 - 8 \div 4 + 7$.

To find the value of this we must do the division first.
Since $8\div4 = 2$ the sum becomes $50 - 2 + 7$ which is 55.

It does not matter what order we work out $50 - 2 + 7$, we get 55 any way.

EXAMPLE 3

Evaluate $18 \div 3 + 4 \times 5$.

Since $18\div3=6$, and $4\times5=20$ we get $18 \div 3 + 4 \times 5 = 6 + 20 = 26$.

EXERCISE 2

Evaluate the following:

a $12 + 8 \div 4$ **b** $80 - 6 \times 7$ **c** $21 \div 7 - 6 \div 3$ **d** $10 + 3 \times 5 - 4 \div 2$

e $100-50\div5$ **f** $3+4\times5-6\div2$ **g** $100-25\times5$ **h** $5+4\div2-1\times3$

TWO PUZZLES

Suppose you are given the 9 symbols: 1, 2, 3, 4, 5, +, -, ×, ÷ and you have to order them make a sum (like the last question in the last exercise).

- Which combination gives the largest possible answer?

- Can you also arrange the symbols to get 0?

BRACKETS

Let us go back to the sum 2 + 3 x 4 (which equals 14).
If we wanted to indicate that the 2 + 3 must be worked out first this can be shown by putting it in brackets:

$$(2+3)x4=20$$

Brackets are always worked out first.

The full convention is: work out **Brackets** first,
then **Multiplication and Division,**
then **Addition and Subtraction.**

EXAMPLE 4

Find $10 \times (15 - 8) + 3$.

We work out the bracket first: $10 \times (15 - 8) + 3 = 10 \times 7 + 3$
then multiply: $= 70 + 3$
then add: $= \underline{73}$.

EXAMPLE 5

Find $7 + 3(18 + 7) - 12 \div 3$.

You may recall that 3(18 + 7) means that the bracket is multiplied by 3.

Brackets first: $7 + 3(18 + 7) - 12 \div 3 = 7 + 3 \times 25 - 12 \div 3$
then × and ÷: $= 7 + 75 - 4$
then + and -: $= \underline{78}$.

EXAMPLE 6

Find $\frac{3(5+9)}{7}$.

Here the 3 is multiplying the bracket and the 7 is dividing the top of the fraction.

Again we work out the bracket first, then multiply.
This gives $\frac{42}{7} = \underline{6}$ as the answer.

EXAMPLE 7

Find $\frac{30+24}{45}$.

There is no bracket in this but the line of the fraction itself acts as a bracket so we always proceed as if there was a bracket on the top (or the bottom) line:

$$\frac{30+24}{45} = \frac{(30+24)}{45} = \frac{54}{45} = \underline{\frac{6}{5}}$$

EXERCISE 3

Evaluate:

a $100 - 3(21 - 9)$

b $8 + 2(3 + 4 - 5)$

c $5(3-1) - 4(2 + 3)$

d $2(80-37) - 48 \div 8$

e $\frac{3(8-3)}{20}$

f $\frac{5(20-9)}{11}$

g $7 + \frac{2(8-5)}{3}$

h $\frac{6(2+8)}{5(8-2)}$

i $\frac{7+15}{11}$

j $\frac{12}{2+1}$

k $\frac{17+13}{17-13}$

l $\frac{2(8-2)}{3(5+7)} + 1$

In the following exercise you are asked to insert brackets in the place which makes the sum correct.

So for example 2 X 3 + 4 - 5 = 9 is not correct.
But 2 X (3 + 4) - 5 = 9 is correct.

EXERCISE 4

Insert a pair of brackets to make the following correct:

a $5 + 4 \times 3 - 2 = 9$

b $3 + 4 - 5 \times 6 = 12$

c $7 \times 6 - 5 + 4 = 11$

d $2 + 3 \times 4 + 5 = 29$

e $5 \times 4 + 3 - 2 = 25$

f $8 \times 6 - 4 \div 2 = 8$

g $2 \times 4 + 6 - 8 = 12$

h $7 + 5 \div 3 + 1 = 5$

i $5 \times 4 + 3 - 2 = 33$

CANCELLING

EXAMPLE 8

Find $\frac{16\times63}{56}$.

Rather than finding 16×63 and then dividing we can cancel any factor in the top with any common factor in the bottom.

Seeing the common factor of 7 in 63 and 56 we cancel it out to get $\frac{16\times9}{8}$.

And we now see a common factor of 8 in the top and bottom which we also cancel out: $\frac{2\times9}{1} = \underline{18}$.

EXAMPLE 9

Find $\frac{15\times27}{18\times24}$.

Looking for common factors we see a common factor of 9 in 27 and 18, so we cancel this out: $\frac{15\times3}{2\times24}$.

Then we see that 3 and 24 have a common factor of 3 which gives: $\frac{15\times1}{2\times8}$

This gives $\underline{\frac{15}{16}}$ as the final answer.

The cancelling could have been done in a different order here, but the answer will be just the same whichever order is chosen.

If we cancel 15 and 18 by 3 first we get $\frac{5\times27}{6\times24}$.

Then we could cancel 27 and 24 by 3: $\frac{5\times9}{6\times8}$.

And finally cancel 6 and 9 by 3: $\frac{5\times3}{2\times8} = \frac{15}{16}$ again.

It is best to cancel by the biggest factor we can find, and keep cancelling as far as possible.

EXERCISE 5

Simplify:

a $\frac{18\times21}{14\times27}$

b $\frac{30\times45}{25\times27}$

c $\frac{36\times50}{30}$

d $\frac{72}{40\times16}$

e $\frac{60\times33\times7}{42\times22}$

f $\frac{3\times54}{18\times20}$

g $\frac{16\times18}{22\times24}$

h $\frac{18\times70}{7\times15\times12}$

i compose a fraction of your own, with at least 4 numbers which cancel down to $\frac{1}{2}$.

SOME REVISION OF DECIMALS

The following exercise is revision of your earlier work on decimals.

EXERCISE 6

Evaluate the following:

a 58.34 + 1.8

b 5.67 + 77.2 + 6

c 88.8 - 3.4

d 3.4516 - 0.222

e 15 - 6.4

f 1.2 + 80.9 - 1.03

g 4.2×3

h 2.006×6

i $5.35 \div 5$

j $0.6 \div 5$

k $23.03 \div 7$

l 3.456×10

m 5.6×100

n 6.3×200

o $33.44 \div 10$

p $0.07 \div 100$

q $16.86 \div 20$

Convert to simple fractions: **r** 0.9 **s** 0.12 **t** 0.011 **u** 3.4

Convert to decimals: **v** $\frac{3}{10}$ **w** $\frac{7}{1000}$ **x** $\frac{19}{100}$ **y** $\frac{25}{10}$

Next we consider multiplying a decimal by a decimal and dividing a decimal by a decimal.

MULTIPLICATION OF DECIMALS

EXAMPLE 10

$0.3 \times 0.07 = \underline{0.021}$

Since $3 \times 7 = 21$, 21 must appear in the answer.
We have only to decide where the decimal point goes.

In terms of fractions the sum is $\frac{3}{10} \times \frac{7}{100}$ which equals $\frac{21}{1000}$.

Changing this back to a decimal we get 0.021.

Notice that in the sum $\underline{0.3 \times 0.07}$ there is a total of 3 figures after the decimal points.
And in the answer $\underline{0.021}$ there are also 3 figures after the decimal point.
In fact this is a general rule:

> In multiplying decimals there are as many figures after the decimal point in the answer as there are after the decimal points in the sum.

EXAMPLE 11

$0.2 \times 0.04 \times 1.2 = \underline{0.0096}$

We find $2 \times 4 \times 12$, which is 96.
We count 4 figures after the decimal points in the sum ($0.2 \times 0.04 \times 1.2$) so there are to be 4 figures after the decimal point in the answer.
We therefore count back 4 places from the end of 96 which means we have to insert two zeros. This gives us 0.0096 which has 4 figures after the point.

EXAMPLE 12

$21.5 \times 0.04 = \underline{0.86}$

Ignoring the points first of all: $215 \times 4 = 860$.
There are 3 figures after the points in the sum so we count back 3 spaces from the end of 860 to get 0.86.

Note that when we get the 860 we do not drop the 0 off immediately but count from the right-hand end of 860.

EXERCISE 7

Multiply the following:

a 0.7×0.8 **b** 0.2×0.3×0.04 **c** 0.007×0.09 **d** 3.3×0.4 **e** 80.8×0.9 **f** 2.7×2.3

g 3.5×0.02 **h** 3.003×0.7 **i** 0.1×0.01 **j** 0.6^2 **k** 4.5^2 **l** 0.4^3

m 3.2×4.3 **n** 0.44×5.6 **o** 5.4^2 **p** 0.34^2

We can use all our earlier multiplication methods on numbers with decimal points in them. We just ignore the points first of all, do the multiplication and insert the point at the end.

EXAMPLE 13

Find 6.3 × 67.

We can use a special method to find 63 × 67 which gives 4221.
And the point goes one place in from the right: <u>422.1</u>.

EXERCISE 8

Multiply the following: ☞

a 4.4 × 4.6 **b** 35 × 3.5 **c** 9.8 × 9.6 **d** 9.7 × 97 **e** 10.3 × 108 **f** 998 × 9.93

g 2.3 × 3.3 **h** 0.21 × 3.4 **i** 333 × 2.3 **j** 3.14 × 25 **k** 3.23 × 67.3 **l** 43.21 × 4.1

DECIMAL DIVISION

EXAMPLE 14

0.18 ÷ 0.003 = <u>60</u>.

If we write the sum as a fraction: $0.18 \div 0.003 = \frac{0.18}{0.003}$.

We can now make the denominator (bottom of the fraction) a whole number by multiplying the numerator and denominator both by 1000.

$\frac{0.18}{0.003} = \frac{180}{3} = 60$.

We divide decimals by writing or thinking of the sum as a fraction, multiplying both numbers so that the denominator is a whole number and then dividing.

EXAMPLE 15

$1.82 \div 0.04 = \frac{182}{4} = \frac{91}{2} = \underline{45.5}$

Here we need to multiply both numbers by 100.

EXAMPLE 16

$\frac{1.6 \times 4.2}{2.4} = \frac{16 \times 42}{240} = \frac{16 \times 7}{40} = \frac{2 \times 7}{5} = \frac{14}{5} = \underline{2.8}$

Here we can multiply top and bottom by 100 as this gives us whole numbers and makes it easy to cancel (because we multiply 1.6 by 10 <u>and</u> 4.2 by 10 to get whole numbers). Then we first cancel by 6 and then by 8.

EXERCISE 9

Find:

a $0.6 \div 0.02$

b $3.6 \div 0.009$

c $0.07 \div 0.1$

d $15 \div 0.03$

e $12.21 \div 0.3$

f $42 \div 0.6$

g $28.21 \div 0.07$

h $6.111 \div 0.9$

i $\frac{2.4 \times 3.9}{2.6}$

j $\frac{0.07 \times 0.8}{0.02}$

k $\frac{5.6 \times 0.12}{0.16}$

l $\frac{4.8}{0.6 \times 0.02}$

17 Special Division

You may recall that it is very easy to divide by 9 in the Vedic system and this is because 9 is close to the base number 10.

EXAMPLE 1

Find 1234 ÷ 9.

$$\begin{array}{r} 9)\underline{1\ \ 2\ \ 3\ \ 4\ \ } \\ \underline{1\ \ 3\ \ 6\,r\,10} \end{array}$$

The answer is 136 remainder of 10, which becomes 137 remainder 1.

EXERCISE 1

Divide the following numbers by 9:

a 222 **b** 602 **c** 37 **d** 2131 **e** 4040 **f** 444

EXAMPLE 2

Find 3172 ÷ 9.

$$\begin{array}{r} 9)\underline{3\quad 1\quad 7\quad 2} \\ \underline{3\quad 4\quad 11\ r\,13} \end{array}$$

Here we find we get an 11 and a 13: the first 1 in the 11 must be carried over to the 4, giving 351, and there is also another 1 in the remainder so we get 352 remainder 4. ☞

A SHORT CUT

However, to avoid the build-up of large numbers like 11 and 13 in this example we may notice, before we put the 4 down, that the next step will give a 2-figure number and so we put 5 down instead:

$$\begin{array}{r} 9)\underline{3\quad 1\quad 7\quad 2} \\ \underline{3\quad 5\quad 2\ r4} \end{array}$$

Then add 5 to 7 to get 12, but as the 1 has already been carried over we only put the 2 down. Finally, 2+2 = 4.

EXAMPLE 3

Find 777 ÷ 9.

$$\begin{array}{r} 9)\underline{7\quad 7\quad 7} \\ \underline{8\quad 6\ r\,3} \end{array}$$

If we put 7 for the first figure we get 14 at the next step, so we put 8.
8+7 = 15 the 1 has already been carried over but if we put the 5 down we see a 2-figure coming in the next step, so we put 6 down.
6+7 = 13 the 1 has been carried over, so just put down the 3.

EXERCISE 2

Divide the following by 9:

a 6153 **b** 3272 **c** 555 **d** 8252

e 661 **f** 4741 **g** 4747 **h** 2938

i 12345 **j** 75057 **k** 443322 **l** 1918161

It is similarly easy to divide by numbers near other base numbers: 100, 1000 etc.

EXAMPLE 4

Suppose we want to divide 235 by 88 (which is close to 100)

We need to know how many times 88 can be taken from 235 and what the remainder is.
Since every 100 must contain an 88 there are clearly 2 88's in 235.
And the remainder will be 2 12's (because 88 is 12 short of 100) plus the 35 in 235. So the answer is <u>2 remainder 59</u>. (24+35=59)

A neat way of doing the division is as follows. ☞

Set the sum out like this: $$\begin{array}{rr|l} 88) & 2 & 35 \\ \hline & & \\ \hline \end{array}$$

We separate the two figures on the right because 88 is close to 100 (which has 2 zeros).

Then since 88 is 12 below 100 we put 12 below 88, as shown below.

$$\begin{array}{rr|l} 88) & 2 & 35 \\ 12 & & 24 \\ \cline{2-3} & 2 & 59 \\ \cline{2-3} \end{array}$$

I.e. 235 ÷ 88 = 2 remainder 59.

We bring down the initial 2 into the answer.

This 2 then multiplies the flagged 12 and the 24 is placed under the 35 as shown.
We then simply add up the last 2 columns.

Note that the deficiency of 88 from 100 is given by the formula *All from 9 and the Last from 10.*

Note also that the position of the vertical line is always determined by the number of noughts in the base number: if the base number has 4 noughts then the vertical line goes 4 digits from the right, and so on.

EXAMPLE 5

Divide 31313 by 7887.
We set the sum out as before: $$\begin{array}{rr|l} 7887) & 3 & 1313 \\ \hline & & \\ \hline \end{array}$$

$$\begin{array}{rr|l} 7887) & 3 & 1313 \\ 2113 & & 6339 \\ \cline{2-3} & 3 & 7652 \\ \cline{2-3} \end{array}$$

Applying *All From 9 and the Last From 10* to 7887 gives 2113.
Bring the first figure, 3, down into the answer.

We now multiply this by the flagged 2113 and put 6339 in the middle row.
Then adding up the last four columns gives the remainder of 7652.

EXERCISE 3

Divide the following (do as many mentally as you can):

a 88)121 **b** 76)211 **c** 83)132 **d** 98)333 ☞

e 887)<u>1223</u>	**f** 867)<u>1513</u>	**g** 779)<u>2222</u>	**h** 765)<u>3001</u>
i 8907)<u>13103</u>	**j** 7999)<u>12321</u>	**k** 7789)<u>21012</u>	**l** 8888)<u>44344</u>

Next we consider the case where the answer consists of more than one digit.

EXAMPLE 6

1108 ÷ 79 = <u>13 remainder 81</u> = <u>14 remainder 2</u>.

We set out the sum marking off two figures on the right, and leave two rows as there are to be two answer figures:

$$\begin{array}{rcc|cc}
79) & 1 & 1 & 0 & 8 \\
 & & & & \\
\hline
 & & & & \\
\hline
\end{array}$$

$$\begin{array}{rcc|cc}
79) & 1 & 1 & 0 & 8 \\
{\scriptstyle 21} & & {\scriptstyle 2} & {\scriptstyle 1} & \\
 & & & {\scriptstyle 6} & {\scriptstyle 3} \\
\hline
 & 1 & 3 & 8 & 1 \\
\hline
\end{array} = \underline{14 \text{ remainder } 2.}$$

Bring the first 1 down into the answer.

Multiply the flagged 21 by this 1 and put the answer (21) as shown in the second row.

Adding in the second column we get 3 which we put down and then multiply the 21 by this 3 to get 63, which we place as shown in the third row.

Add up the last two columns, but since the remainder, 81, is greater than the divisor, 79, there is another 79 contained in 81 so there are 14 79's in 1108 with 2 remaining.

EXAMPLE 7

1121123 ÷ 8989.

$$\begin{array}{rccc|cccc}
8989) & 1 & 1 & 2 & 1 & 1 & 2 & 3 \\
{\scriptstyle 1011} & & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 1} & & \\
 & & & {\scriptstyle 2} & {\scriptstyle 0} & {\scriptstyle 2} & {\scriptstyle 2} & \\
 & & & & {\scriptstyle 4} & {\scriptstyle 0} & {\scriptstyle 4} & {\scriptstyle 4} \\
\hline
 & 1 & 2 & 4 & 6 & 4 & 8 & 7 \\
\hline
\end{array}$$

The initial 1 comes down into the answer and multiplies the flagged 1011.

This is placed as shown in the second row.

Adding in the second column we put 2 down in the answer and then multiply the 1011 by it. Put 2022 in the third row.

Adding in the third column we get 4 which we put down and also multiply by 1011.

So we put 4044 in the fourth row and then add up the last four columns to get the remainder.

> Once the vertical line has been drawn in you can see the number of lines of working needed: this is the number of figures to the left of this line (3 figures and therefore 3 lines of working in the example above).

EXERCISE 4

Divide the following:

a 89)1021 **b** 88) 1122 **c** 79)1001 **d** 88)2111 **e** 97) 1111

f 888) 10011 **g** 887)11243 **h** 899)21212 **i** 988)30125 **j** 8899)201020

DIVISOR ABOVE A BASE NUMBER

A very similar method, but under the formula *Transpose and Apply* allows us to divide numbers which are above a base number.

EXAMPLE 8

1489 ÷ 123 = 12 remainder 13

Here we see that 123 is close to the base of 100 so we mark 2 figures off on the right.
In fact the method is just as before except that we write the flagged numbers as bar numbers:

$$\begin{array}{rrr|rr}
123) & 1 & 4 & 8 & 9 \\
\bar{2}\,\bar{3} & & \bar{2} & \bar{3} & \\
 & & & \bar{4} & \bar{6} \\
\cline{2-5}
 & 1 & 2 & 1 & 3 \\
\cline{2-5}
\end{array}$$

Bring the initial 1 down into the answer.
Multiply this 1 by the flagged $\overline{23}$ and write $\bar{2}, \bar{3}$.
Add in the second column and put down 2.
Multiply this 2 by the $\overline{23}$ and put $\bar{4}, \bar{6}$.
Then add up the last two columns.

EXERCISE 5

Divide the following:

a $123)\underline{1377}$ **b** $131)\underline{1481}$ **c** $121)\underline{256}$ **d** $132)\underline{1366}$

e $1212)\underline{13545}$ **f** $161)\underline{1781}$ **g** $1003)\underline{321987}$ **h** $111)\underline{79999}$

Two other variations, where negative numbers come into the answer or remainder are worth noting.

EXAMPLE 9

Find 10121 ÷ 113.

$$\begin{array}{rrrr|rr}
1\;1\;3) & 1 & 0 & 1 & 2 & 1\\
\bar{1}\;\bar{3} & & \bar{1} & \bar{3} & & \\
 & & & 1 & 3 & \\
 & & & & 1 & 3\\ \hline
 & 1 & \bar{1} & \bar{1} & 6 & 4
\end{array} = \underline{89 \text{ rem } 64}.$$

When we come to the second column we find we have to bring $\bar{1}$ down into the answer, multiplying this by the flagged $\overline{13}$ means we add 13 in the third row (two minuses make a plus).

The answer $1\bar{1}\bar{1}$ we finally arrive at is the same as 100 - 11 which is 89.

EXAMPLE 10

Find 2211 ÷ 112.

$$\begin{array}{rrr|rr}
1\;1\;2) & 2 & 2 & 1 & 1\\
\bar{1}\;\bar{2} & & \bar{2} & \bar{4} & \\
 & & & 0 & 0\\ \hline
 & 2 & 0 & \bar{3} & 1
\end{array} = 20 \text{ rem } \overline{29} \text{ or } \underline{19 \text{ rem } 83}.$$

20 remainder -29 means that 2211 is 29 short of 20 112's.
This means there are only 19 112's in 2211, so **we add 112 to -29** to get 19 remainder 83.

EXERCISE 6

Divide the following:

a 112)$\underline{1234}$

b 121)$\underline{3993}$

c 103)$\underline{432}$

d 1012)$\underline{21312}$

e 122)$\underline{3333}$

f 123)$\underline{2584}$

g 113)$\underline{13696}$

h 1212)$\underline{137987}$

i 111)$\underline{79999}$

j 121)$\underline{2652}$

k 1231)$\underline{33033}$

18 Percentage Changes

In this chapter we will be using some of our earlier special types of multiplication to find out what a certain quantity becomes when it is increased, or reduced, by a certain percentage.

EXAMPLE 1

Increase 30 by 50%.

This means increase 30 by a half of 30.
And since half of 30 is 15 we increase 30 by 15 to get 45.

EXAMPLE 2

Reduce 16 by 25%.

This means take 25% of 16 away from 16.
Since 25% is a quarter and a quarter of 16 is 4, we take 4 from 16 to get 12.

EXERCISE 1

Increase: **a** 20 by 50% **b** 44 by 50% **c** 1000 by 50% **d** 40 by 25%

e 8 by 25% **f** 30 by 10% **g** 40 by 20% **h** 24 by 100%

Reduce: **i** 30 by 50% **j** 200 by 50% **k** 12 by 25% **l** 80 by 25%

m 32 by 25% **n** 80 by 10% **o** 500 by 1% **p** 600 by 2%

INCREASING BY 10%

You will recall that there is a quick of multiplying a number by 11.

EXAMPLE 3

Find **a** 34 × 11 **b** 77 × 11 **c** 567 × 11.

a 34 × 11 = <u>374</u> We simply put 7, the total of 3 and 4, between the 3 and the 4.

b 77 × 11 = <u>847</u> Here the total is 14 and we carry 1 to the left.

c 357 × 11 = Here 8 (the total of 3 and 5) and 12 (the total of 5 and 7) is placed between the 3 and the 7. When the 1 is carried we get an answer of <u>3927</u>.

EXAMPLE 4

Find 54 × 1.1.

Multiplying by 1.1 is the same as multiplying by 11 except that the decimal point is put one place to the left: 54 × 11 = 594, therefore 54 × 1.1 = <u>59.4</u>.

EXAMPLE 5

Increase 32 by 10%.

We increase a number by 10% by multiplying it by 1.1. So we simply find 32 × 1.1.

32 × 1.1 = <u>35.2</u>.

We increase a number by 10% by multiplying it by 1.1.

EXERCISE 2

Multiply by 11:

a 26 **b** 41 **c** 53 **d** 67 **e** 84 **f** 38 **g** 88 **h** 73

i 333 **j** 345 **k** 712 **l** 2314 **m** 339 **n** 709 **o** 852 **p** 888

Multiply by 1.1:

q 42 **r** 35 **s** 72 **t** 66 **u** 234

Increase by 10%:

v 24 **w** 17 **x** 74 **y** 232 **z** 258

PERCENTAGE INCREASES

In a similar way we increase by 1%, 2% etc. by multiplying by 1.01, 1.02 etc. And we know an easy way to multiply by numbers like 101, 102 ...
So let us first revise this type of multiplication.

EXAMPLE 6

Find **a** 32 × 101 **b** 32 × 102 **c** 76 × 102 **d** 222 × 103.

a 32 × 101 = <u>3232</u>.

b 32 × 102 = <u>3264</u>. We put twice 32 on the right.

c 76 × 102 = $76_{1}52$ = <u>7752</u>. Twice 76 is 152 so we carry the 1.

d 222 × 103 = $222_{6}66$ = <u>22866</u>. We can have only 2 figures on the right when multiplying by 103.

EXERCISE 3

Find:

a 67 × 101 **b** 98 × 101 **c** 44 × 102 **d** 36 × 102 **e** 49 × 102 **f** 57 × 102

g 88 × 102 **h** 36 × 103 **i** 14 × 106 **j** 333 × 103 **k** 321 × 104 **l** 26 × 104

> We increase a number by 1%, 2%, 3% . . .
> by multiplying it by 1.01, 1.02, 1.03 . . .

EXAMPLE 7

Increase 43 by 2%.

Increasing by 2% is the same as multiplying by 1.02.
And we multiply by 1.02 by multiplying by 102 and placing the decimal point 2 places to the left.
So 43 × 1.02 = <u>43.86</u>.

EXERCISE 4

Increase: **a** 23 by 3% **b** 41 by 2% **c** 88 by 1% **d** 34 by 2%

e 14 by 4%
f 8 by 1%
g 19 by 3%
h 34 by 3%
i 64 by 3%
j 26 by 7%
k 222 by 2%
l 123 by 3%
m 50 by 5%
n 55 by 6%

PERCENTAGE REDUCTIONS

Next we consider how to multiply a number by numbers just under 1, instead of just over 1. We want to be able to multiply easily by numbers like 0.99, 0.98, 0.97.
So first we look at multiplying by 99, 98, 97 etc.

EXAMPLE 8

Find 23 × 99.

Remember that $99 = 10\bar{1}$. So $23 \times 99 = 23 \times 10\bar{1} = 23\overline{23} = \underline{2277}$.

You can see that the answer, 2277, is formed from the 23 by first reducing it by 1 to get 22 (using *By One Less than the One Before)*, and then applying *All from 9 and the Last from 10* to the 23 to get 77.

EXAMPLE 9

Find 23× 98.

$23 \times 98 = 23 \times 10\bar{2} = 23\overline{46} = \underline{2254}$. We double 23 and apply *All from 9* . . . to 46 because 98 is 2 below 100.

EXAMPLE 10

Find 23 × 94.

$23 \times 94 = 23 \times 10\bar{6} = 23_{\bar{1}}\overline{38} = \underline{2162}$. Here we have to carry $\bar{1}$ to the left.

EXERCISE 5

Find:

a 67 × 99
b 48 × 99
c 39 × 99
d 19 × 99
e 91 × 99
f 44 × 98
g 73 × 98
h 98 × 98
i 56 × 98
j 71 × 98
k 33 × 97
l 44 × 97
m 16 × 97
n 82 × 97
o 8 × 97
p 55 × 96
q 22 × 96
r 33 × 93
s 41 × 92
t 34 × 91

Now we can make percentage reductions.

To reduce by 1%, 2%, 3% . . .
we multiply by 0.99, 0.98, 0.97 . . .

That is, we multiply by $1.0\bar{1}$, $1.0\bar{2}$, $1.0\bar{3}$ . . .

EXAMPLE 11

Reduce 56 by 1%.

We find 56 × 0.99 and since 56 × 99 = 5544
therefore 56 × 0.99 = <u>55.44</u>.

EXAMPLE 12

Reduce 56 by 3%.

We find 56 × 0.97 and as $56 \times 97 = 56_{\bar{1}}\overline{68} = 5432$
therefore 56 × 97 = <u>54.32</u>.

EXERCISE 6

Reduce:

a 33 by 1%
b 77 by 1%
c 83 by 1%
d 43 by 2%
e 26 by 2%
f 78 by 2%
g 32 by 3%
h 32 by 5%
i 58 by 3%
j 47 by 4%
k 7 by 3%
l 21 by 7%
m 59 by 4%
n 11 by 9%
o 28 by 22%

19 Transformations

Something is transformed when it is changed in some way.
We will be considering four types of mathematical transformation:

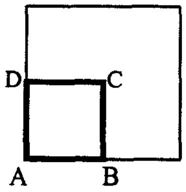
Enlargement, **Reflection**, **Rotation**, and **Translation**.

ENLARGEMENT

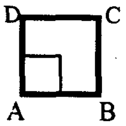
If we take a square as an illustration we can see that there are different ways of enlarging it.

EXAMPLE 1

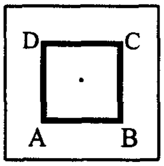
a



b



c



In both **a** and **c** above we start with a square ABCD and a larger square has been obtained from it. In both cases the larger square is double in size to the first square. We say that **the scale factor of the enlargement is 2.**

The difference between **a** and **c** is that in **a** the enlargement is **from the point A**, whereas in **c** the enlargement is **from the centre of the first square**.
We say that **the centre of enlargement is at A** in **a** and at the centre of ABCD in **c**.

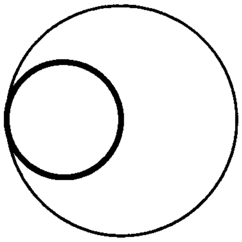
An enlargement is specified by **the centre of enlargement**
and **the scale factor of the enlargement**.

Comparing **a** and **b** above note that the centre of enlargement is at A in both cases. The difference is that in **b** the original square has been reduced in size: its sides are in fact half those of the original square. The scale factor in this case is ½.

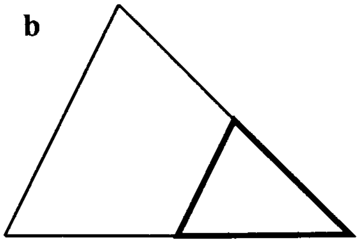
EXERCISE 1

In each diagram below the bold shape is being transformed. Copy the diagram, put a dot at the centre of enlargement and write down the scale factor of the enlargement:

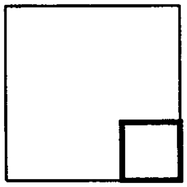
a



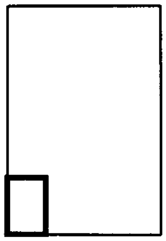
b



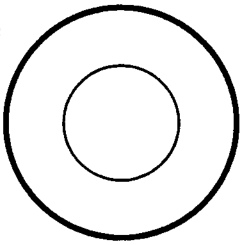
c



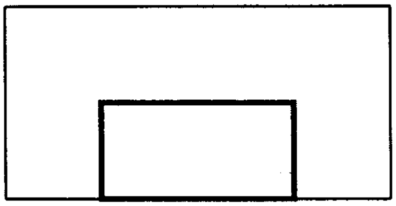
d



e



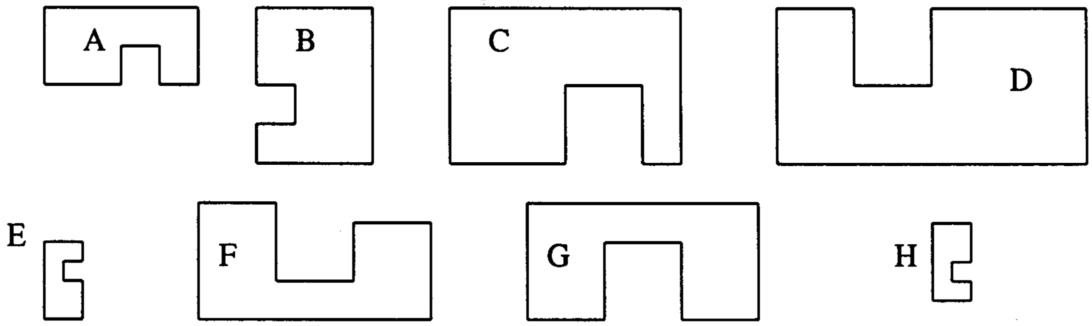
f



Note that in any kind of enlargement the shape of the original figure is not changed by the enlargement (a square is still a square when it is enlarged), only the size is altered.
Note also that if a figure is reduced in size we still call it an enlargement, but the scab factor will always then be a number less than 1.

EXERCISE 2

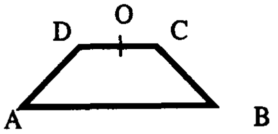
Which shapes are enlargements of shape A:

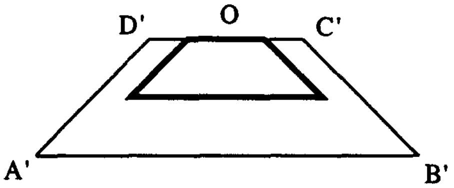


We may also be given a starting shape, a centre of enlargement and a scale factor and be asked to enlarge the starting shape.

EXAMPLE 2

Enlarge the trapezium ABCD by scale factor 2, using the mid-point of CD as centre of enlargement.





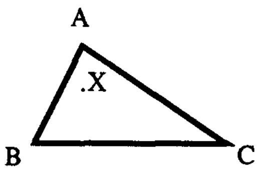
Since the scale factor of the enlargement is 2 each of the corners A,B,C and D must be twice as far from the centre of enlargement than in the original shape. So ABCD has been enlarged to A'B'C'D'.

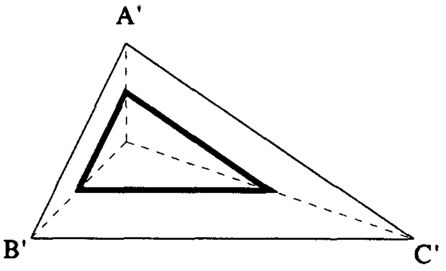
- Measure the distances of A, B, C and D from the centre of enlargement, O, and check that these are doubled in the enlargement.

Enlargements come under the *Proportionately* Sutra of Vedic Mathematics.

EXAMPLE 3

Enlarge the triangle ABC shown using X as the centre of enlargement and a scale factor of 2.





We draw a line from X to A and since the scale factor is 2 we extend this line until it is twice its original length. This gives us the point A' shown on the right above.

Similarly we extend the lines XB and XC to get B' and C' and then complete the triangle.

1 On a sheet of graph paper number 0 to 8 on the x-axis and 0 to 5 on the y-axis.
Draw the triangle with vertices at (1,2), (5,2) and (2,4).
Using (2,3) as a centre of enlargement and a scale factor of 2 draw an enlargement of the triangle.

EXERCISE 3

For each of the enlargements below start with a rectangle ABCD which has base AB = 2cm and height BC = 1cm (use squared paper and label the enlargements A'B'C'D'):

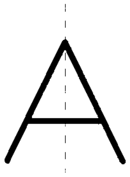
	Centre of Enlargement	Scale Factor
a	A	3
b	D	2
c	centre of rectangle	2
d	mid-point of AB	3
e	mid-point of BC	2
f	C	½
g	1cm to the left of A	2

For each of the enlargements below start with an equilateral triangle ABC where BC is the base and BC = 2cm (use triangular spotty paper and label the enlarged triangles A'B'C');

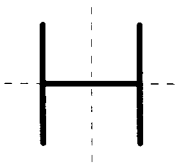
	Centre of Enlargement	Scale Factor
h	A	3
i	centre of triangle	½
j	centre of triangle	2
k	B	1½
l	mid-point of BC	3
m	1cm above A	2

REFLECTION

Some shapes are symmetrical. For example the letter A has a symmetry: we could draw a vertical line through it and each half would be a reflection of the other half



Similarly the letter H has two lines of symmetry- one vertical and one horizontal.

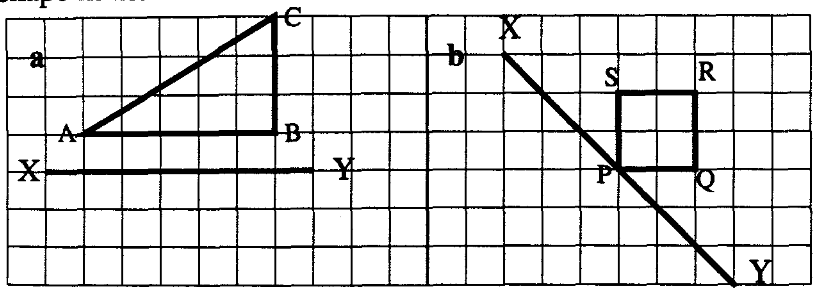


2 Write out the letters C D E M T W X Y Z and draw in the lines of symmetry.

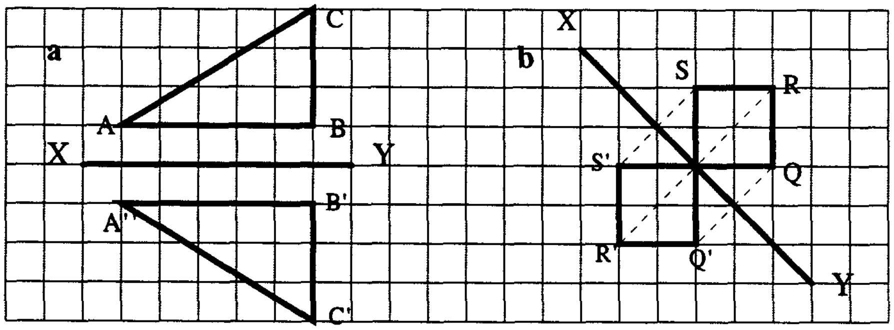
We reflect a shape when we are given one half of the shape and the position of the line of symmetry. We use the *Transpose and Apply* formula.

EXAMPLE 4

Reflect the shape in the line XY.

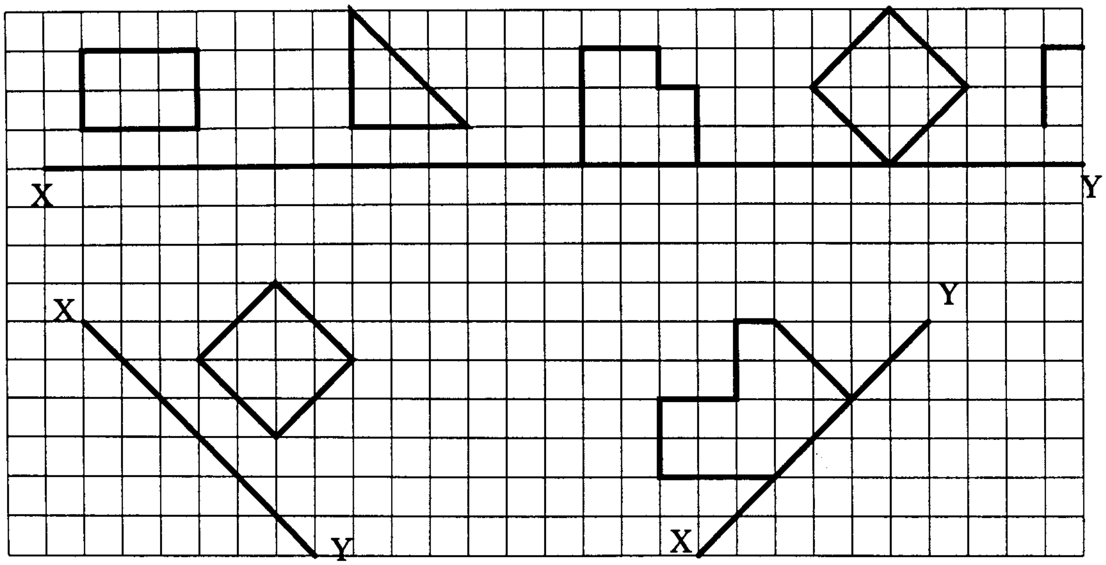


The reflection is always as far behind the mirror as the object is in front of it and so we can find the reflection of each of the corner points of the shape and join them up.
So the reflection of A is 1 unit below the line XY, and so is the reflection of B.
C is 4 units above the line so we mark a point, C, 4 units below it. Then we can complete the triangle.
In part b we must be more careful. P is on the line and so it will remain there. How far is Q from the line XY?
It is 1 square diagonal from the line, and so we measure a further diagonal on the other side of XY and mark a point Q. Similarly for S.
R is 2 diagonals from XY so we extend 2 further diagonals on the other side of XY and then join up the square.



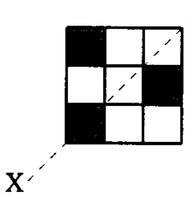
EXERCISE 4

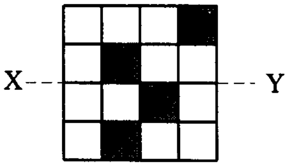
Copy the following shapes onto square dotty paper and reflect them in the line XY:

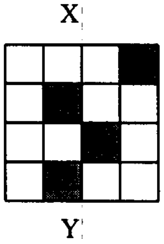


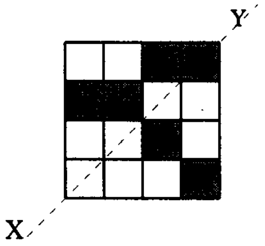
EXERCISE 5

Copy the following shapes and shade in the fewest number of extra squares that will make the shape symmetrical about XY:







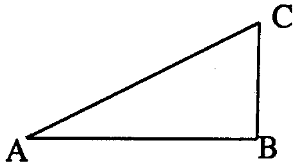


ROTATION

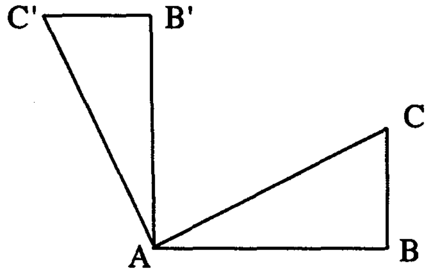
EXAMPLE 5

Another way in which objects can be transformed is by rotating them.

Rotate the triangle below 90° anticlockwise about A.



We have to imagine the whole triangle turning anticlockwise around A.
So A is fixed and the line AB will eventually be vertical.



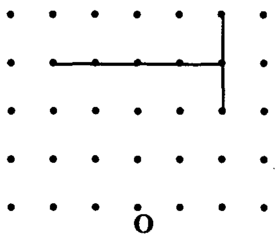
We see that three things need to be specified in order to carry out a rotation, in addition to the object itself which is being rotated:

To rotate an object we need to know:
1) the centre of the rotation
2) the direction of rotation (clockwise or anticlockwise)
3) the angle of rotation.

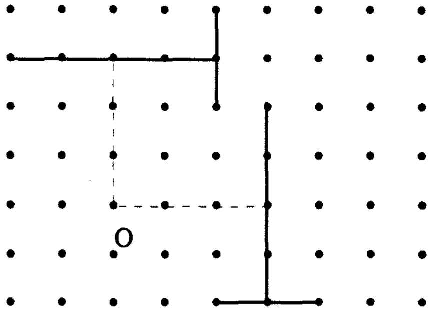
The one exception to this is 180° rotations which do not need a direction because the result is the same whether you rotate clockwise or anticlockwise.

EXAMPLE 6

Rotate the shape below 90° clockwise about the point marked O.



It may not be so obvious where the rotated shape will go here.
But we may note that the point in the middle of the horizontal line is 3 units vertically up from O, and so it will rotate to the point 3 units to the right of O.
It is then not difficult to draw in the rest of the figure.



Note that with rotations the shape and the size of the object being rotated are not affected by the rotation, only the position is affected.

Rotations come under the Sutras *By Alternate Elimination and Retention,* and *By Mere Observation* because we usually mentally rotate a part of a figure (a point or line) first, ignoring the rest of the shape, and then add the rest of it.

For more difficult rotations or as a check tracing paper can be used. Place the tracing paper over the shape to be rotated and trace it. Then put your pen or pencil on the centre of rotation and keep it there while you turn the paper by the angle given. You will then see where the rotated shape goes.

EXERCISE 6

Use square spotty paper and carry out the rotations below.
The object for each of these is a triangle like the one in Example 5 but with a base of 4cm and a height of 1 cm. Label your rotated shape A'B'C'.

	Centre of Rotation	Direction of Rotation	Angle of Rotation
a	A	clockwise	90°
b	B	clockwise	90°
c	B	anticlockwise	90°
d	C	-	180°
e	mid-point of AB	anticlockwise	90°
f	1cm below A	clockwise	90°
g	2cm above mid-point of AB	clockwise	90°
h	1cm above and 1cm to the right of C	-	180°

EXERCISE 7

Use triangular spotty paper for the following:

a Draw an equilateral triangle ABC, where the base AB is 2cm.
Rotate this 60° clockwise about A.
Now rotate both triangles, as a single figure, 120° clockwise about C.

b Draw a regular hexagon ABCDEF of side 2cm, where AB is the base and the vertices are labelled in an anticlockwise direction.
Rotate the figure 180° about D.
Also rotate both shapes, as a single figure, 120° clockwise about A.

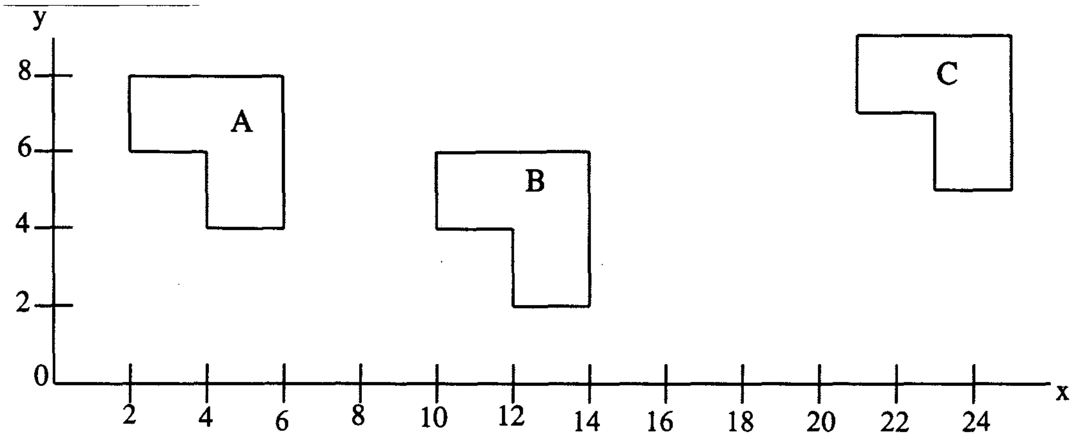
☞

c Draw a rhombus ABCD of side 1cm, where AB is the base and vertices labelled in an anticlockwise direction again.
Rotate ABCD 120° clockwise about C and also 120° anticlockwise about C.
Rotate the whole shape thus formed **i** 120° clockwise about A.
ii 120° anticlockwise about A.

TRANSLATION

A translation occurs when a shape is moved from one position to another without turning it or changing it in any other way.

EXAMPLE 7



In this graph shape A has been translated to B.

To describe a translation precisely we only need to specify how much the original shape has been moved horizontally, and how much it has been moved vertically.

Since shape A has moved 8 units to the right and 2 units down we say that the translation is $\begin{bmatrix} 8 \\ -2 \end{bmatrix}$ where the 8 is the horizontal movement and -2 indicates 2 units downwards.

Similarly to translate shape C to B the translation would be $\begin{bmatrix} -11 \\ -3 \end{bmatrix}$ because we have to move C 11 units to the left and 3 units down.

The top number in a translation is the horizontal part of the translation
and
the bottom number is the vertical part.
Translations are positive to the right and positive upwards,

They come under the *By Addition and By Subtraction* formula.

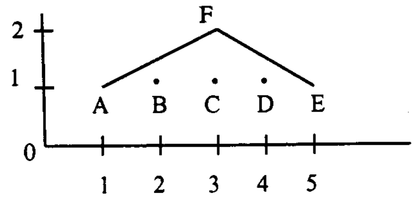
3 What translation moves C to A? 4 What translation moves B to A? 5 What translation moves A to C?

EXERCISE 8

a The square A with coordinates (2,2), (4,2), (4,4), (2,4) is translated by $\begin{bmatrix} -4 \\ -3 \end{bmatrix}$.

What will be the coordinates of the square after the translation?
What translation will move the square back to A?

b



This figure consists of 3 dots and 2 lines.
What will be the coordinates of A, B, C, D, E and F after the translation $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$?

c A pentagon, A, has coordinates (0,0), (-2,-2), (-4,-2), (-6,0), (-3,2).
Draw the pentagon, A, and translate it, by $\begin{bmatrix} 10 \\ 4 \end{bmatrix}$ to B.

Then translate the pentagon, A, by $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ to get shape C.

What translation will move B to C?

We can also combine the various transformations together, as shown in the next

EXERCISE 9

You will need graph or squared paper:

a Number your x-axis from 0 to 9 and your y-axis from 0 to 7.
Draw the rectangle A(1,2), B(4,2), C(4,4), D(1,4). Label this P.
Reflect P in the line y=2. Label this Q.
Rotate P 90° clockwise about C. Label this R.
Translate P by $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$. Label this S.
Enlarge P with C as centre of enlargement and scale factor ½. Label this T.

b Using ½cm for 1 unit on each axis, number the x-axis from -6 to 28 and the y-axis from 0 to 16.
Plot the rectangle (0,0), (2,0), (2,1), (0,1).
i Rotate the rectangle 90° clockwise about the bottom right corner of the rectangle,
ii Enlarge the rotated rectangle, scale factor 2, with centre at the bottom right of the rectangle.
Now go back to instruction **i** and repeat instructions **i** and **ii** as many times as you can.

c Label axes from 0 to 9 on the x-axis and 0 to 7 on the y-axis.
Plot the right-angled triangle (1,1), (3,1), (3,2).
Rotate the triangle 90° clockwise about (3,1). Label this P.
Reflect P in the line y=3. Label this Q.
Rotate Q 90° clockwise about (5,3). Label this R.
Reflect R in the line x=6. Label this S.
Describe the rotation that transforms S back onto the original triangle.

20 Constructions

In ancient Greek times geometrical constructions could only be properly carried out using two geometrical instruments: a straight edge and compasses.

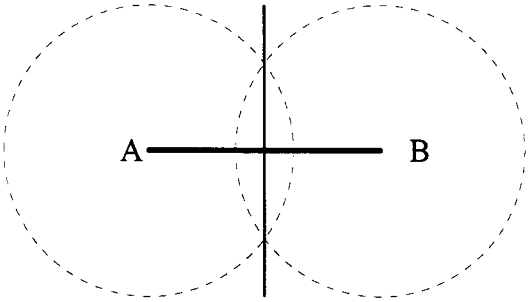
This was because the straight line and the circle were considered to be fundamental forms from which all other forms could be created.

Today we have other instruments like protractors, set squares etc. but in this chapter we will use only ruler and compasses. Plain paper should be used for the work in this chapter.

BISECTING A LINE

Suppose we have a straight line, drawn with our ruler, and that we want to cut the line in half (bisect it).

● Draw a horizontal line about 6cm long (the length is not important) with some space above ands on each side of it.

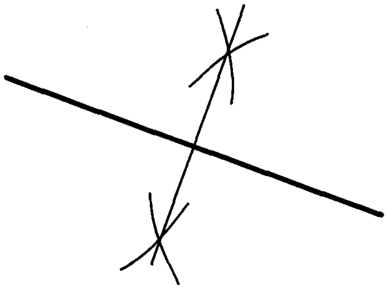


Suppose the line is AB in the diagram.
Put your compass point on A and open your compasses until the pencil is over half of the distance from A to B.
Draw a circle, as shown.
Without altering the compass setting put the point on B and draw another circle. Join the points of intersection of the two circles.
This line bisects the original line.

It is usual for all construction lines to be drawn faintly on the paper with only the required result being drawn in boldly at the end.

EXERCISE 1

a Draw an oblique (slanting) line and find the centre of it using this method. Note that it is not essential to draw full circles, you could just draw arcs where you expect the intersections to be, as shown below.

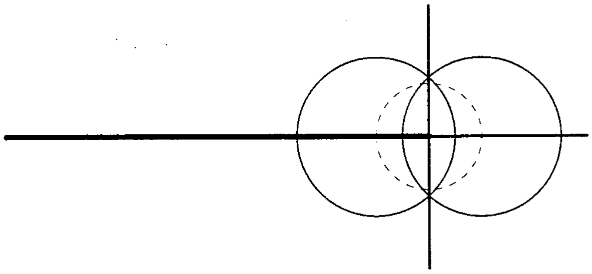


b Construct a triangle with sides 3, 6 and 7cm.
Bisect each of the three sides of the triangle. If you are accurate you should find that the three lines meet at a single point. This point is called the circumcentre.

c The circumcentre is the centre of a circle which goes through all three vertices of the triangle. Put your compass point on the circumcentre and draw a circle which passes through the three vertices of the triangle. This is called the circumcircle.

This simple bisection construction gives us more than just the mid-point of the line, it also gives us four right angles where the two lines intersect. The constructed line is therefore called the perpendicular bisector.

1 Draw a line about 6cm long. Suppose that we want to construct a right angle at each end of the line.



Starting at the right end of the line (shown in bold above) first extend the line to the right and draw a circle (shown dashed) centred on the end of the line.

Use the two points where this circle cuts the horizontal line as centres to draw two more, larger, circles which intersect. The vertical line can then be drawn in.
Finish the construction at the right end of your line and then make a right-angle at the left-hand end of your line by the same method.

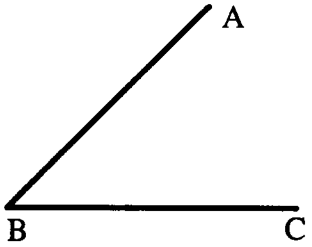
EXERCISE 2

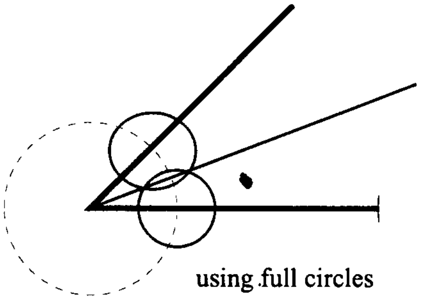
a Construct a triangle with sides 4, 5 and 6cm.
Bisect the three sides and hence draw in the circumcircle.

b Draw an oblique line 8cm long.
Construct a right-angle at each end of the line and complete a rectangle of height 5cm.

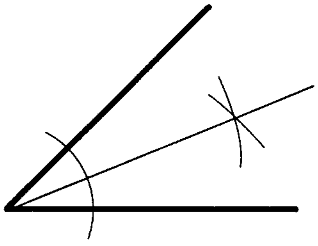
BISECTING AN ANGLE

Suppose we have the angle ABC and that we want to draw a line which bisects the angle.





using full circles



using arcs only

We first draw a circle centred on the corner B (shown dashed).
This circle cuts the two arms of the angle in two points. We use these points as centres for two more circles which must be the same size as each other and which must intersect.
The bisector can then be drawn from the corner B, through these intersection points and on.

EXERCISE 3

In this exercise, except for part **a**, you may use full circles in the constructions or only arcs.

a Draw two angles like the one above and bisect them using the two methods shown above.

b Now draw an obtuse angle and bisect it in the same way.

c Construct a triangle with sides 3, 6 and 7cm.
Bisect all three of the angles of your triangle.

The three angle bisectors of a triangle always meet in a single point.
Check that your bisectors meet in one point, or very nearly.
This point is called the **incentre** of the triangle.

From the incentre it is possible to draw a circle that touches all three sides of the triangle.

d Put your compass point on your incentre and draw the **incircle**, just touching all the sides without going outside the triangle.

- A line which just touches a circle without going inside it is called a tangent.

CONSTRUCTING ANGLES

Bisecting angles allows us to construct various angles without using a protractor.

EXERCISE 4

a Construct a right angle by the method shown earlier.
Bisect the right-angle. What angle have you formed? (How many degrees?)

b Draw a base line AB of length 5cm.
Draw a circle of radius 5cm centred on A and another centred on B.
Complete the equilateral triangle.
Bisect the angle at A. What angle have you formed?

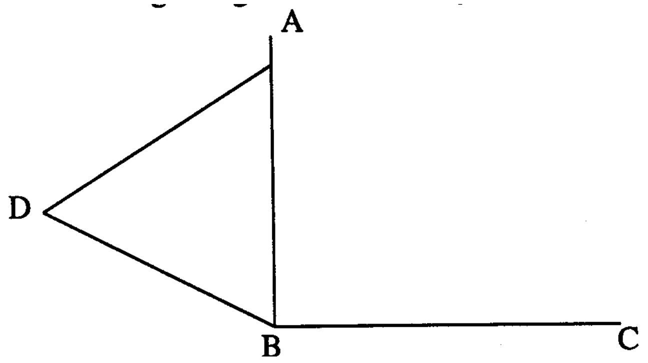
c Bisect the angle at B in your equilateral triangle. Call the two halves p (below) and q (above).
Bisect angle p. What angle have you formed?

EXAMPLE 1

Construct an angle of 105°.

One way is to construct angles of 90° and 15° which add up to 105.

To do this we could construct a right-angle and then an equilateral triangle:



2 Construct the right-angle ABC using only ruler and compasses.
Construct an equilateral triangle on one arm of the right-angle as shown above.

Now bisect angle B of the triangle and bisect the upper part again so that the 15° angle you get is next to the line AB. Mark clearly your 105° angle.

3 In the above diagram what size is the angle DBC?

EXERCISE 5

a Construct a triangle with sides 4, 5 and 6cm.
Bisect all three angles to find the incentre and draw the incircle.

b Construct an angle of 120° by adding two 60° angles.

c Construct an angle of 120° by adding 30° to 90°.

d Construct an angle of 135°.

e An angle of 105° can also be constructed by adding angles of 45° and 60°.
Construct the 105° angle in this way.

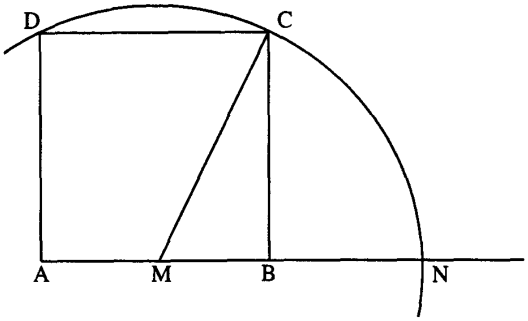
THE GOLDEN RECTANGLE

There is a certain shape of rectangle which has many remarkable properties.
The ratio of its sides is called the golden ratio because the ratio arises in all sorts of ways and in many different areas of study.
The golden rectangle can be constructed as follows.

EXERCISE 6

a Using ruler and compasses construct a square ABCD of side about 8cm. Be sure to have at least 5cm of space to the right of it. Extend AB about 5cm.

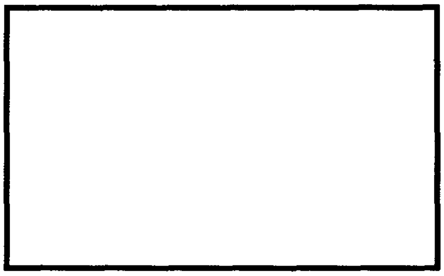
b Bisect the base to find the mid-point, M, and join M to the top right vertex of the square.



c Draw a circle (or part of a circle) with M as centre and MC as radius. This gives you the point N on the base line.

d N, A and D are three corners of the golden rectangle. So construct a perpendicular at N and extend the line DC to meet this perpendicular.

e Draw over your golden rectangle to make it stand out.



This rectangle is said to have the most pleasing shape and artists and architects have used the it in their designs.

Next you can produce another golden rectangle by copying the one you have just made, as explained in the next exercise.

EXERCISE 7

a First draw a base line faintly about 15cm long with some space above it.

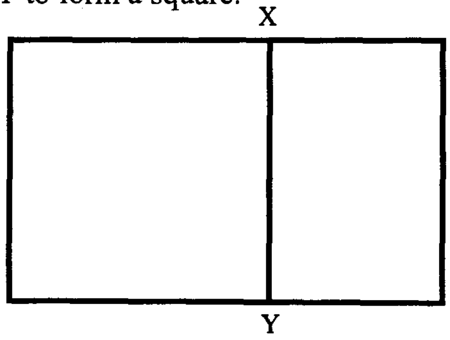
b Measure the side AN by putting your compass point on A and opening it until the pencil is on N exactly. Put the point on one end of your base line and mark off the distance on the other end. Draw in the base of your golden rectangle.

c Now measure the height of your rectangle in the same way. Put your compass point on each end of your new construction and make an arc above it where you expect the top corners to be.

d Measure a diagonal of your golden rectangle and transfer it to the new construction. That means put the point at one base corner and make an arc where you expect the opposite corner to be. This should cut the previous mark. Transfer both diagonals in this way and then complete the golden rectangle.

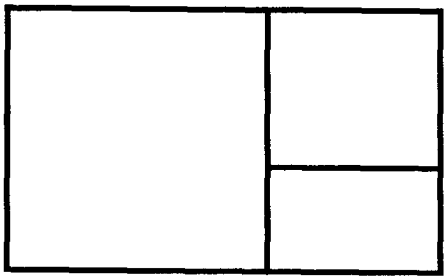
Next we will use this diagram to illustrate another beautiful property of the golden rectangle.

e We want to form a square on the left side of the rectangle. Measure the height of your rectangle with your compasses and mark off this distance from both left corners to get the points X and Y. Join XY to form a square.



It is a remarkable fact that the smaller rectangle produced in this way is also a golden rectangle.

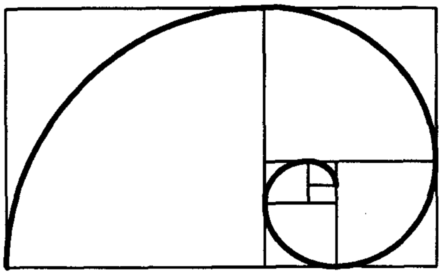
f Now we repeat this process by constructing a square in the top part of the smaller golden rectangle. Measure the top of the rectangle with compasses and mark off the distance on the long sides.



g The small rectangle produced is again a golden rectangle. Repeat this again by forming a square on the right of the smallest rectangle, and then again by forming a square on the bottom of the smallest rectangle.

h If you can see how the squares you have been drawing spiral round in a clockwise direction you may be able to draw in some more golden rectangles. Do as many as you can.

i it is now possible to draw in the golden spiral:



The spiral goes on spiralling in towards a central point.
It is also possible to extend the spiral outwards by adding a square on the bottom and then on the right and so on indefinitely.

THE PENTAGRAM

First we construct a pentagon as follows:

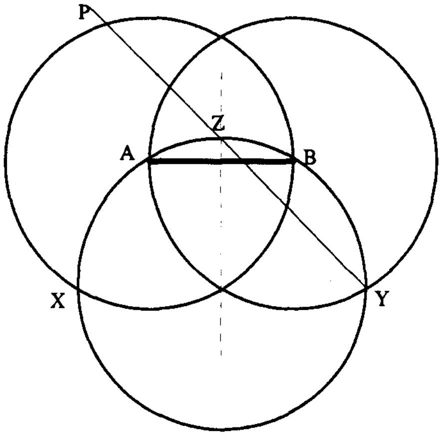
EXERCISE 8

a Draw a base line AB about 6cm long in the middle of your page.

b From each end of the line draw a circle with radius equal to the length of the base line.

c At the lower point of intersection of these circles draw another circle the same size as the first two. The two lower points where this circle meets the other two circles we call X and Y.

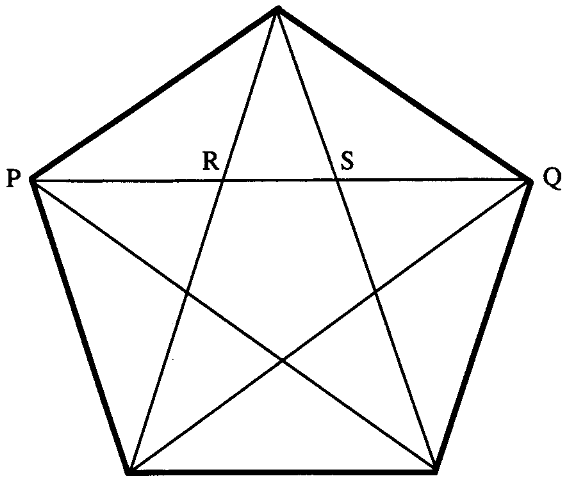
d Bisect the line AB. Where this bisector meets the top of the lower circle we call Z.



e Next draw a line from Y to Z and extend it until it meets the top left circle at P. Similarly extend the line XZ until it meets the top right circle at Q.

f Join AP and BQ. These are two more sides of the pentagon.

g To find the top corner of the pentagon take your compasses, still set at the radius of the circles and with the point on P make an arc where you expect the top corner of the pentagon to be. Make a similar mark from Q. Where they intersect is the top corner so you can now complete the pentagon.



h Finally you can draw in the pentagram by joining all five diagonals of the pentagon in bold or with a colour.

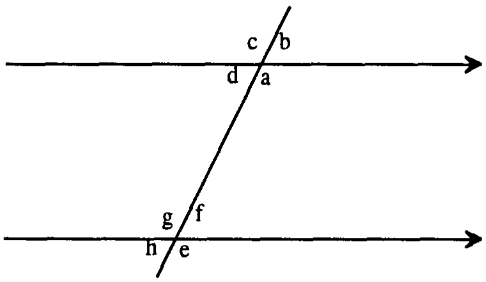
Note that each diagonal cuts each other diagonal at two points. So PQ for example is cut at R and at S.

The pentagram is full of golden ratios: PR:RS is a golden ratio,
and so is QS:SR
and PQ:PS and PQ:RQ.

Similarly for all the other diagonals.

21 Bearings

PARALLEL LINES



The arrows on the two lines above means that they are exactly parallel.

The line cutting across the parallel lines produces the eight angles a, b, h.

- Some of these angles are equal: which do you think are equal?

Angles b and d will be equal because b and c add up to 180° and so do d and c.
Angles which are equal in this way are called vertically opposite angles.

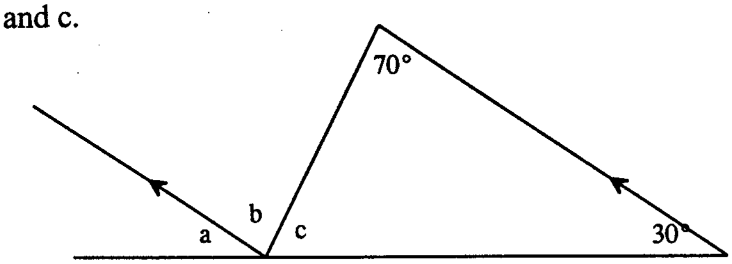
Similarly a = c, e = g and h = f.
This means that a, c, a and g are all equal and that b, d, f and h are all equal.

Angles like a and a which form an F shape are equal (we can imagine the angle sliding down the oblique line until it fits exactly at e): we will call these F-angles.

Angles like d and f, which form a Z shape are therefore also equal: we will be referring to Z-angles.
c and g are also F-angles and a and g are also Z-angles

EXAMPLE 1

Find angles a, b and c.



Here we see a pair of parallel lines and so we can look for vertically opposite angles and F and Z-angles.

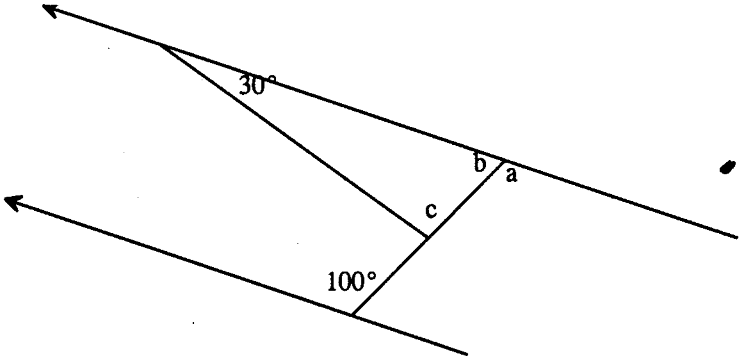
There are no vertically opposite angles in the diagram, but there is a pair of F-angles and a pair of Z-angles.

b = 70° because b and the 70° angle are Z-angles,
and a = 30° because a and the 30° angle are F-angles.

Since a+b+c = 180° we can then get c, so c = 80°.

EXAMPLE 2

Find angles a,b and c.

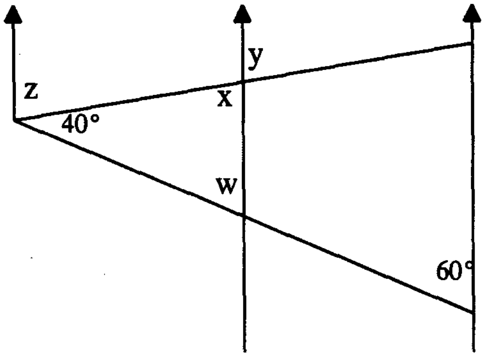


Here we may notice a pair of Z-angles.

a = 100° because a and the 100° angle are Z-angles, and since a+b = 180°, b = 80°.
Since the angles in a triangle total 180° we then get c = 70°.

EXAMPLE 3

Find w, x, y, z.

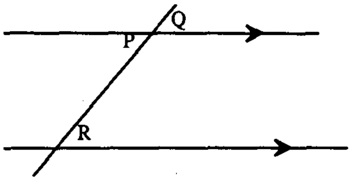


Here we have 3 parallel lines.
There are 2 pairs of F-angles, a pair of Z-angles and a pair of vertically opposite angles.

w = 60° (they are F-angles),
therefore x = 80° (three angles in a triangle),
x=y (vertically opposite) so y = 80°,
and z=x (they are Z-angles) so z = 80°.

This is not the only way of solving this.

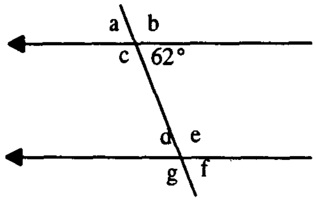
We can summarise these results about angles in parallel lines as follows:

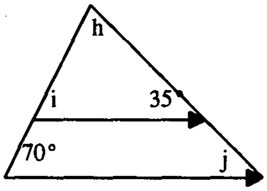


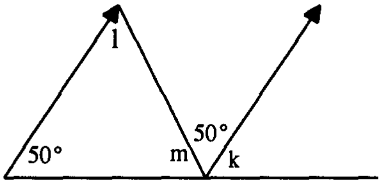
P=Q as these are vertically opposite,
Q=R as these are F-angles,
P=R as these are Z-angles.

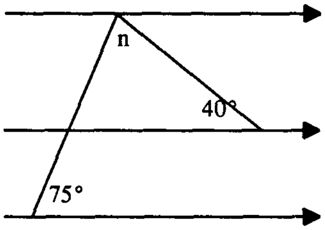
EXERCISE 1

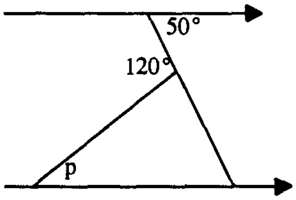
Find the angles marked with letters:

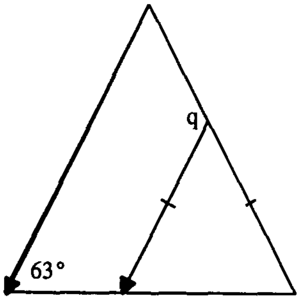












SCALE DRAWING

Plans and maps are very useful where large structures are to be studied or built. These are examples of scale drawings where the shape of every detail is the same as in the original but the size is much smaller. This is very similar to enlargements dealt with in the chapter before the last, where the scale factor is less than 1.

EXAMPLE 4

A rectanglar lawn measures 45m by 30m. By drawing a plan of the lawn find the distance between opposite corners of the lawn.

We need to select a suitable scale: if we use lm = l cm the rectangle will still be rather large. So we may choose 5m = 1 cm which will make the rectangle 9cm by 6cm.

- Draw this rectangle on squared paper. Write the scale: 5m=1 cm next to it.

- Measure the length of a diagonal: you should get about 10.8cm (anything between 10.7 and 10.9 is acceptable).

Since 1 cm is equivalent to 5m we must multiply the 11 cm by 5 to get an answer of about 55m.

The scale of 5m = l cm can also be written as 500:1 because 5m=500cm and so the lawn size has been reduced by a scale factor of 500.

EXAMPLE 5

A triangular plot of land measures 200m by 144m by 120m. Find the distance from the mid-point of the longest side to the opposite corner.

Again we must first decide on a suitable scale.
If we choose 20m = 1 cm then we have to divide the three given sides by 20. This means drawing a triangle with sides 10cm by 7.2cm by 6cm.

1 Construct this triangle (you will need to use compasses). Write the scale next to it.

Join the mid-point of the longest side to the opposite corner and measure this distance. You should get 4.3 or 4.4cm for this.

Then using the scale again we convert back to metres by multiplying this by 20. This gives a final answer of 86 or 88m.

2 Write the scale 20m = 1 cm in ratio form as shown at the end of Example 4.

3 Using a suitable scale make a careful plan of your classroom.

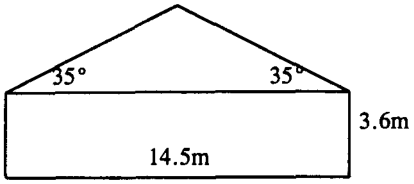
EXERCISE 2

Give your answers to the following to 2 significant figures.

a Find the length of the diagonal of a square of side 300m.

b The triangle ABC has a right angle at A, AB=70cm and BC=lm. The bisector of angle B meets the opposite side at X. Find BX.

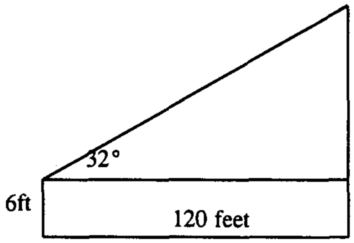
c The diagram shows the end view of a bungalow. Find the total height.



d Using a scale of 20:1 construct the rectangle WXYZ where WX=1.6m and XY=50cm. Find the distance of the mid point of WZ from X.

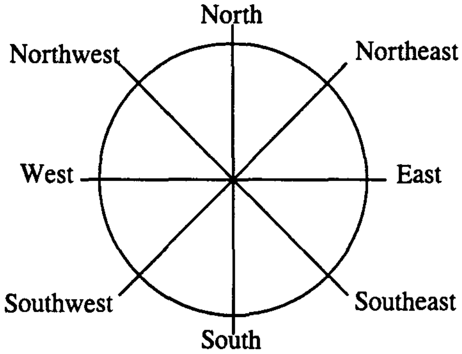
e In the triangle PQR PQ=16mm, QR=30mm and PR=34mm. Draw an enlarged version of this triangle. The bisector of P meets the bisector of Q at S. Find SR.

f A man 6ft tall finds the angle between the horizontal direction and the direction of the top of a building to be 32°. If he is standing 120 feet from the base of the building find the height of the building (see diagram).



BEARINGS

You will be familiar with the directions of the compass: North, South, East and West.



There are other directions as shown in the diagram above.
Northwest, for example, is exactly half way between the north and west directions. This means there is an angle of 45° between each of these directions.

> **A bearing is a direction or angle measured clockwise from the North direction. We always suppose the North direction to be vertically upwards on our page.**

So, for example, if you travelled northeast this could be described by the bearing 045° (bearings are always given with 3 figures) because northeast is halfway between north and east.

And a bearing of 135° would be the same as a direction southeast.

EXERCISE 3

Convert the following directions to bearings (angles):

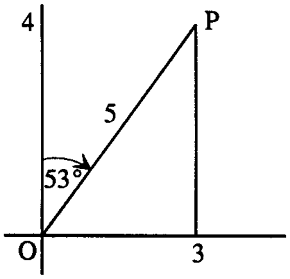
a east **b** south **c** southwest

d west **e** north **f** northwest

Convert the following bearings to compass directions: **g** 090° **h** 225° **i** 315°

USING BEARINGS

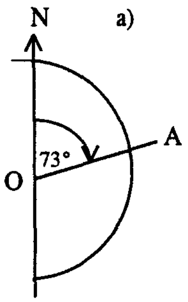
The position of a point, relative to some other point, can be described in various ways.
It could be described, for example, by a pair of coordinates: P(3,4) means go 3 units to the right from O and then 4 units up.

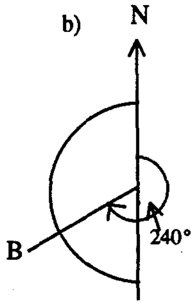


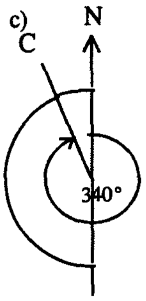
But this position could also be described by giving the bearing of P from O and its distance from O. We could say P has a bearing of 053° and is 5 units from O.

EXAMPLE 6

Show the position of **a** a point A which is on a bearing of 073° from a point O,
b a point B on a bearing of 240° from a point O and
c a point C on a bearing of 340° froma point O.







In each case begin by marking the point, O, in the middle of the page and drawing the north line in.

a Place your protractor with the centre at O and the base line of the protractor along the north line. See the first diagram above.

Measure clockwise from the north to find 73° and make a mark there.
Join this to O and write the angle as shown.

b If you have not got a circular protractor then for B and C you will have to lay the protractor on the other side, as shown.

Since 180° brings you round to the south direction you need another 60° to get 240°.
So measure 60° clockwise from the south and mark B.

c Similarly for C you can measure 160° from the south, or better still is to measure 20° backwards from the north because 340° is 20° short of the full 360°.

EXERCISE 4

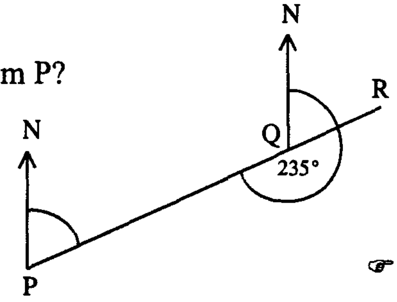
Draw a diagram showing the following bearings:

a 30° **b** 145° **c** 200° **d** 260° **e** 310° **f** 95°

EXAMPLE 7

If the bearing of P from Q is 235° what is the bearing of Q
It is not necessary to make an accurate drawing for this. A sketch is sufficient.

om P?



The sketch shows the positions of P and Q. By drawing a north line at P we can then see that the bearing of Q from P is the angle marked.
Next if we extend the line PQ to say R then we can see that angle NQR is the same as the required angle (they are F-angles). And we can see from the way the 235° has been split up that we can get the answer by taking 180° from 235° giving an answer of <u>55°</u>.

In the following questions draw two north lines each time and look for F and Z-angles.

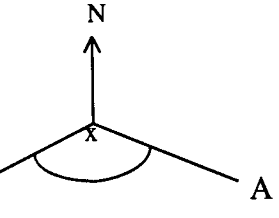
EXERCISE 5

Find the bearing of Y from X given that the bearing of X from Y is

a 290° **b** 195° **c** 160° **d** 75° **e** 330° **f** 133°

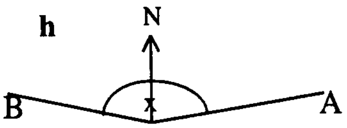
Find the size of the angle marked x in each of the following sketches:

g



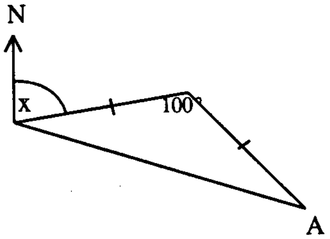
Bearing of A is 105°,
bearing of B is 230°.

h



Bearing of A is 75°,
bearing of B is 290°.

i



Bearing of A is 100°.

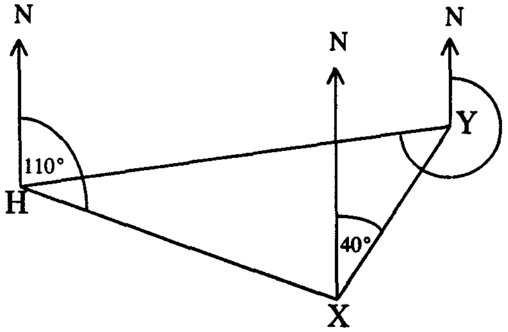
To find the bearing of Y from X, given the bearing of X from Y, there is a simple rule we can use: If the angle is less than 180° we add 180°;
and if the angle is greater than 180° we subtract 180°.

- Check your answers to a to f by this method.

EXAMPLE 8

A ship sails for 10 km on a bearing of 110° from a harbour H to a point X, and then sails for 7km on a bearing of 040° to a point Y.
By means of a scale drawing find the bearing of H from Y and the distance the ship must travel to return directly to the harbour. ☞

First we mark the harbour H, and then draw in a north line going upwards.
Then using a protractor we can draw the 110° bearing as shown below.



Next we must choose a suitable scale. Say 1cm = 2km. This means we measure 5cm along the line. This gives us the position of X.
Next we draw a north line at X and draw a bearing of 040°. According to our scale this line should be 3½cm long.

The bearing of H from Y is found by drawing in a north line at Y and measuring the angle shown. This should be 262°.

The distance required is HY, and by measurement this is 7.0cm, which according to our scale is 14.0km.

EXERCISE 6

a A ship A is due east of a lighthouse and 20km from it. Another ship B is 40km from the lighthouse and on a bearing of 051° from it. Using a scale of 1cm = 5km make a scale drawing and find the distance of B from A and the bearing of B from A.

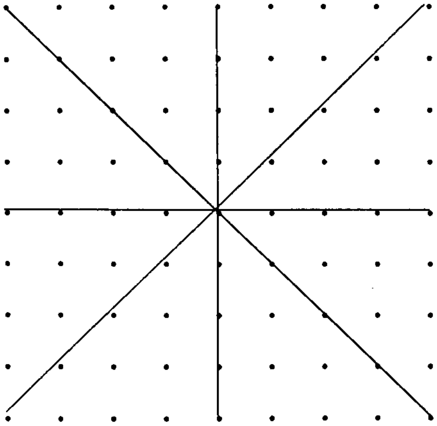
b Three villages X, Y, Z have bearings of 020°, 130° and 230° from a point O respectively.
Their distances from O are 20km, 120km and 100km respectively.
Make an accurate scale drawing and find the distances XY, XZ and YZ.
Find also, by measurement, the bearing of Y from X, Z from Y and Z from X.

c A helicopter flies from base B on a bearing of 082° for 200km and then changes direction to fly on a bearing of 120° for 180km. By making an accurate drawing find his distance from B and the bearing he must take to return to base.

d ABC is an equilateral triangle with base AB=80m and B due east of A.
Find the bearing of C from A and the bearing of C from B.
Use a scale drawing to find the distance and bearing of the mid-point of BC from A.

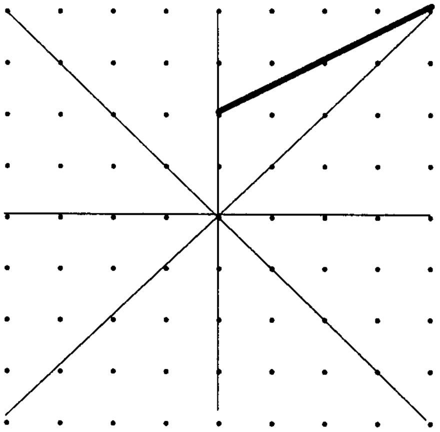
RANGOLI PATTERNS

- On square spotty paper draw the following lines:



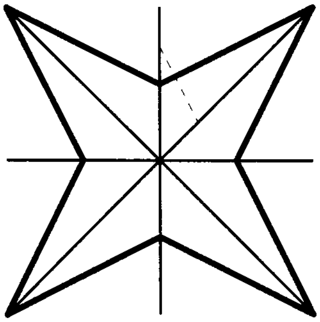
This is the base for a Rangoli pattern.
The four lines are four lines of symmetry.

We start with a line of our choice. Suppose we choose the one below:



We then have to reflect the line in all the lines of symmetry.

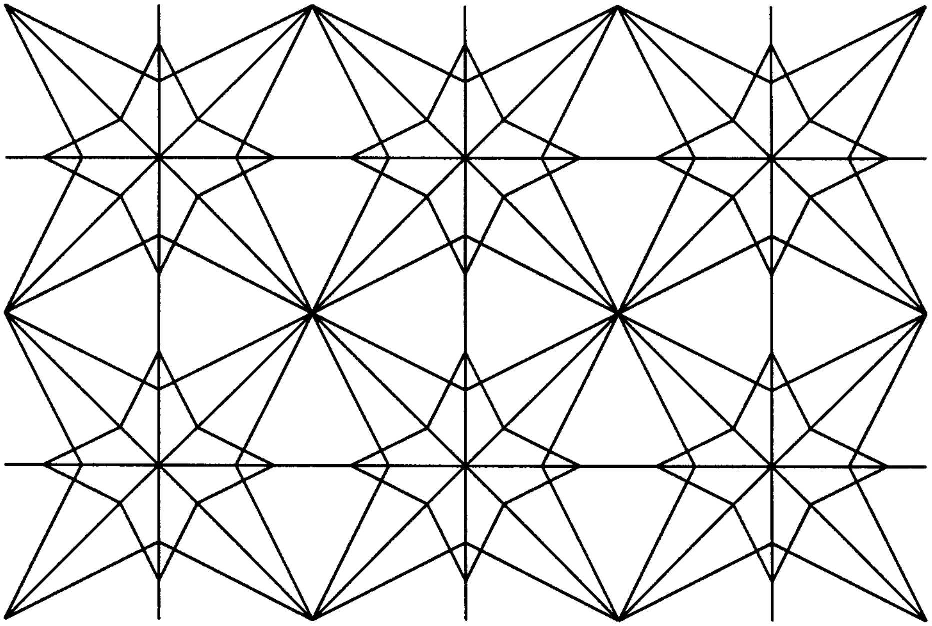
❋ **First reflect the line in the vertical axis, then in the diagonal that goes from top left to bottom right, then in the horizontal axis and so on until after 7 reflections the following pattern is complete:**



Now you can add another line and reflect it in all the axes of symmetry. For example the line shown dashed in the diagram above.

- ● Add the line shown to your diagram and reflect it in the lines of symmetry.

Now, since this figure fits in a square four of them will tessellate and so you could combine several of these together:



- ● Make a Rangoli pattern of your own, tessellate this pattern and colour it in.

Many Islamic designs are based on tessellated Rangoli patterns.

22 Recurring Decimals

We have seen that we can convert a fraction to a decimal by dividing the numerator by the denominator.
This is easy enough if the denominator is a small number.
If the denominator is large however we may use the Sutra *By One More than the One Before.*

EXAMPLE 1

Convert the fraction $\frac{1}{19}$ to its decimal form.

By One More than the One Before means one more than the number before the 9 in this fraction. Since there is a 1 before the 9, one more than this is **2**.
So for this fraction the Sutra says ***By 2***.

We call **2 the Ekadhika** # (the number *one more*) and we keep dividing by this **2** (rather than dividing by 19).

So we start with 0 and a decimal point, then **2** (the Ekadhika) into 1 (the numerator) goes 0 remainder 1:

$$\frac{1}{19} = 0.{}_{1}0$$

Note carefully that we put the remainder before the answer, 0. ☞

`Ekadhika' is pronounced with two long syllables followed by two short ones, the long syllables being twice the length of the short ones.

We now have 10 in front of us and we divide this by **2**: $\frac{1}{19} = 0.{}_{1}0\ 5$
We then divide this 5 by **2**.

This gives 2 remainder 1 so we now have: $\frac{1}{19} = 0.{}_{1}0\ 5{}_{1}2$
Again we put the remainder, 1, before the 2.

At every step we divide the last answer figure by the Ekadhika,
2, and put the answer down as the next answer figure.
Any remainder is put before that answer figure.

Then we divide 12 by **2** and put down 6.

$$\frac{1}{19} = 0.{}_{1}0\ 5{}_{1}2\ 6\ 3{}_{1}1{}_{1}5{}_{1}7{}_{1}8\ 9\ \ldots.$$

- Study the answer above and make sure that you agree with all the numbers.

1 Continue the division until you have at least 20 figures after the decimal point.

You should find that after 18 figures the answer figures are starting to repeat themselves: you will get 10 to divide and then 5, which is how you started the sum.
This means that those 18 figures will repeat themselves forever.

To show that the 18 figures repeat endlessly we put a dot over the first and last figures. These dots indicate the first and last numbers of the block which repeats indefinitely:

$$\frac{1}{19} = 0.{}_1\,\dot{0}\,5{}_1 2 6 3{}_1 1{}_1 5{}_1 7{}_1 8 9{}_1 4 7{}_1 3{}_1 6 8 4 2\,\dot{1}\ .$$

The answer is 0. 0 5 2 6 3 1 5 7 8 9 4 7 3 6 8 4 2 1: the remainder figures are not part of the answer.

Dividing by 2 like this is very much easier than dividing by 19 of course!

Convert $\frac{11}{19}$ to a recurring decimal.

The Ekadhika is still 2 because we still have 19 in the denominator.

But we begin by dividing 2 into 11 (the numerator): $\frac{11}{19} = 0.{}_1 5$

2 into 11 goes 5 remainder 1.
Next we divide 2 into 15: $\frac{11}{19} = 0.{}_1 5{}_1 7$

2 Continue this division until it starts to repeat and then put a dot on the first and last figures.

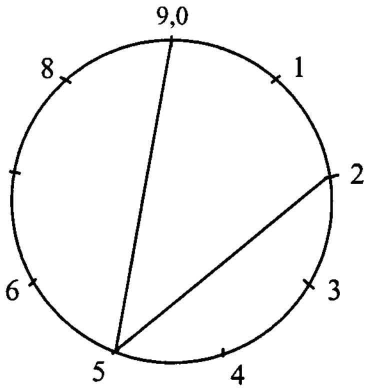
There are two important things which you may have noticed as you worked out $\frac{11}{19}$.
It also has 18 figures recurring, like $\frac{1}{19}$.
And in fact the figures are the same as $\frac{1}{19}$ but they just start in a different place.

We can show this cycle of 18 figures as a pattern on the 9-point circle.

RECURRING DECIMAL PATTERNS

On Worksheet 4 you will have some 9-point circles.

3 Write the recurring decimal for $\frac{1}{19}$ below the first circle.
To plot the pattern put your pencil at 0 (the first figure of the decimal) and draw a line to 5 (the second figure of the decimal).
Then from 5 draw a line to 2 (the third figure of the decimal).



Continue to join the points as shown by the figures in the recurring decimal.
When you get to the 1 at the end you have one more line to draw: because the decimal repeats in a cycle of 18 figures the 1 at the end is followed by the first figure which is 0.
So join 1 to 0 and you have finished.

You should have a diagram which is symmetrical about a vertical axis.
We will not draw the pattern for $\frac{11}{19}$ because it is really the same cycle and so will look identical, when it is finished, to the one you have drawn. If you are not convinced however you can draw it on the next circle.

A DIFFERENT DENOMINATOR

EXAMPLE 3

Convert $\frac{17}{29}$ to a decimal.

Here the number before the 9 is 2 so *One More than the One Before* means one more than 2, which is **3**.
So **the Ekadhika is now 3**.

This means we start by dividing 17 (the numerator) by **3**, and keep on dividing by **3**: **3** into 17 goes 5 remainder 2.

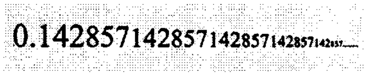
$$\frac{17}{29} = 0.{}_{2}5$$

Then **3** into 25 goes 8 remainder 1; and **3** into 18 goes 6 and so on.

4 Find the full recurring decimal for 29 .

❋ You should find that there are 28 figures in this recurring decimal. Plot the pattern for $\frac{17}{29}$.

A SHORT CUT



You have now worked out three recurring decimals: $\frac{1}{19}$, $\frac{11}{19}$ and $\frac{17}{29}$.
The first two have 18 recurring figures and the third has 28.

$$\begin{array}{lccccccccc} \frac{1}{19}=0.{}_1 & \dot{0} & 5 & {}_12 & 6 & 3 & {}_11 & {}_15 & {}_17 & {}_18 \\ & 9 & {}_14 & 7 & {}_13 & {}_16 & 8 & 4 & 2 & \dot{1} \end{array}$$

❋ Write out the three recurring decimals again but this time write $\frac{1}{19}$ and $\frac{11}{19}$ in two rows of 9 figures ($\frac{1}{19}$ is shown above). Include the remainders and be careful to line the bottom row up with the figures of the top row. Write the third recurring decimal with two rows of 14 figures.

5 If you look carefully at these three decimals you may notice something. Write down what you find.

You should have found that in each case the total of every column is the same: every column adds up to 9. This means that when we have got half way through a sum we can write down the second half from the first half- we just take every figure in the first half from 9 and this gives us the second half!

Now you may ask "how do I know when I am half way through?" and there is a simple way to find out. Look at the fraction $\frac{1}{19}$. Take the numerator from the denominator: 19 - 1 = 18.

When 18 comes up you are half way through. In your recurring decimal for $\frac{1}{19}$ you should see 18 at the end of the first line.

Similarly for $\frac{11}{19}$, since 19-11 = 8 you should find 8 at the end of your first line.
And for $\frac{17}{29}$, 29-17 = 12 (it is written ${}_1$ 2 and it comes up after 14 figures).

If the difference of numerator and denominator comes up in a recurring decimal you are half way through and you can get the second half by taking all the figures in the first half from 9.

The difference of numerator and denominator does not always appear however.

EXAMPLE 4

Convert $\frac{9}{39}$ to a decimal.

First we note that the Ekadhika is **4** now (1 more than 3); we write **E = 4.**
Also the half way number is 30 (39-9=30): we write **H = 30.**

We begin by dividing the Ekadhika, **4**, into the numerator, 9: $\frac{9}{39} = 0.{}_12\ 3{}_30$

The first three steps give us 12, 3 and 30 and so we find that the half way number has come up after just 3 figures.

So we just take each of the first three figures from 9 to get the full answer:

$$\frac{9}{39} = 0.{}_1\dot{2}\ 3{}_30\,7\,6\ \dot{9}$$

❋ Calculate $\frac{9}{39}$ again but without using the half way number- just keep dividing by 4. You should, of course, get the same answer as before.

6 Find the recurring decimal for $\frac{10}{39}$. (E=**4** and H=29)
You should get a 6-figure recurring decimal for this just as you did in Example 4, but in this case the half way number does not come up.

❋ Plot the patterns for $\frac{9}{39}$ and $\frac{10}{39}$ on 9-point circles.

EXERCISE 1

Find the recurring decimal for:

a $\frac{25}{29}$ **b** $\frac{24}{39}$ **c** $\frac{29}{39}$ **d** $\frac{3}{49}$ **e** $\frac{44}{69}$ **f** $\frac{44}{79}$

g $\frac{1}{99}$ **h** $\frac{1}{9}$ (E will be 1 here because there is nothing before the 9)

i plot your answers from **b** to **f** above on 9-point circles.

PROPORTIONATELY

So far all our fractions have had 9 as the last figure of the denominator.
In fact the special method we have been using can only be applied when the denominator ends in 9. However there are various other devices we can apply so that the method will work with other denominators.

One of these is to use the *Proportionately* formula.

EXAMPLE 5

Find the recurring decimal for $\frac{7}{13}$.

The denominator does not end in 9 here, it ends in 3.
However we know we can multiply the top and bottom of a fraction by any number we like without changing its value.

Can we multiply top and bottom of $\frac{7}{13}$ by a number that gives us a denominator ending in 9?

We can multiply top and bottom by 3 which gives $\frac{7}{13} = \frac{21}{39}$.
And we now find the decimal for $\frac{21}{39}$ exactly as before:

$$\frac{21}{39} = 0.{}_1\dot{5}{}_3 3{}_1 8\ 4\ 6\ \dot{1}$$

- Check that you agree with this answer (the half way number is 18, and it comes up after 3 figures).

7 What will $\frac{1}{7}$ need to be multiplied by on the top and bottom so that the denominator ends in 9?

In the next exercise each fraction will need to be multiplied so that it ends in a 9 in the denominator.

EXERCISE 2

Convert to recurring decimals:

a $\frac{1}{7}$ **b** $\frac{2}{13}$ **c** $\frac{5}{23}$ **d** $\frac{17}{33}$ **e** $\frac{9}{11}$ **f** $\frac{3}{17}$

EXERCISE 3

Find correct to 4 decimal places (you will need to find the first 5 figures and then give the answer to 4 d.p.):

a $\frac{18}{59}$ **b** $\frac{67}{89}$ **c** $\frac{100}{109}$ **d** $1\frac{3}{7}$ **e** $\frac{20}{13}$ **f** $\frac{99}{49}$

We will take this subject further in Chapter 17.

23 Formulae

A formula is an expression of some principle or law.
We have sometimes referred to the Sutras and Sub-Sutras of Vedic Mathematics as formulae.

A very simple example of a formula is $I = 12f$.
This is a formula for converting a number of feet, f, into inches, I.
However many feet there are we multiply this by 12 to get the number of inches because there are 12 inches in a foot.

All algebraic expressions and formulae are composed of three types of symbols. There are constants, like the 12 in this formula, variables, like the I and the f, and there are operators, like the multiplication in the formula (the 12 is multiplying f).

Sometimes the constant may be denoted by a letter and for this the letters at the beginning of the alphabet are usually used.

Variables are usually denoted by letters from the end of the alphabet, but not always because sometimes it is convenient to use the first letter of the name of the variable. So we may use I for inches and f for feet because this helps us the understand the formula.

Einstein's famous formula $\mathbf{E = mc^2}$ gives the energy, **E**, equivalent to a quantity of matter, **m**.
c is the velocity of light, which is a constant equal to about 300,000,000 metres per second.
So in $\mathbf{E = mc^2}$ c is a constant, E and m are variables and the operators are multiplication and squaring.

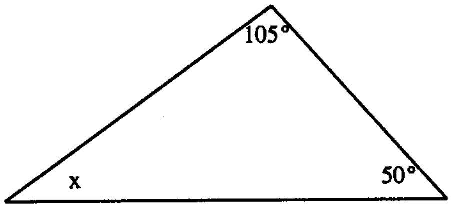
This remarkable formula shows that a small amount of matter can yield a huge amount of energy because the mass of the matter is multiplied by c^2.

1 What number is c^2 equal to?

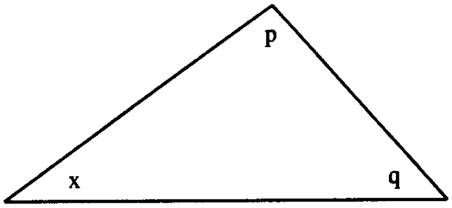
EXAMPLE 1

Find a formula for finding the third angle of a triangle when two angles are known.

Let us start by taking a particular example:
We would find x here by adding the two known angles and subtracting the total from 180. We get x=25°.



So if we had p and q instead of 105 and 50 we would add p and q, which is (p+q), and subtract this from 180.
So we get $\underline{x = 180 - (p+q)}$.



If you substitute the values of 105 and 50 for p and q in this formula you will get the same answer as before.

2 Would it be wrong to write the answer as $x =- p + q$?
Is this the same as the underlines answer above?

Using specific values in a problem to help decide what to do with letters is a useful way of going about solving problems with letters. This is an application of the Sutra *Specific and General.*

EXAMPLE 2

Find a formula for solving equations like $\frac{ax}{b} = c$.

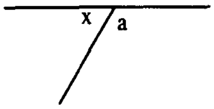
Here a, b and c stand for numbers, so one such equation would be $\frac{2x}{3} = 4$.
If we solve this equation we will then see what to do with the one which has a, b and c instead of numbers.
You will recall that we multiply the 4 by the 3 and then divide by the 2.
So x = 6 would be the answer.

Now we do the same thing with $\frac{ax}{b} = c$.
Multiplying b by c we get ax = bc. Then dividing bc by a we get $\underline{x = \frac{bc}{a}}$ (since we cannot actually divide bc by a we write $\frac{bc}{a}$).

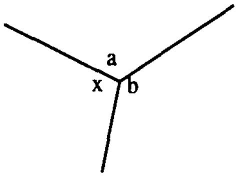
EXERCISE 1

Find a formula for the angle x in each of the following:

a



b



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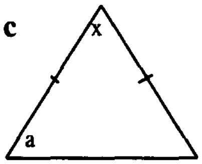

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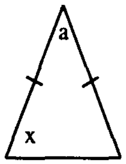


e A rectangle has sides b and h and its area is A. Write down a formula for A. f A rectangle has sides b and h. Write down a formula for its perimeter, P. Find formulae for solving the following equations:

g $x + a = b$ **h** $ax = b$ **i** $ax + b = c$ **j** $\frac{x}{a} + b = c$

EXAMPLE 3

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}.$$

Another way in which a formula may be useful is illustrated here.
Given two fractions, $\frac{a}{b}$ and $\frac{c}{d}$ we add them by cross-multiplying and adding for the numerator of the answer, and we put the product of the denominators as the denominator.
The formula expresses the method of adding fractions much more neatly than words do.

EXAMPLE 4

$ax^n \times bx^m = abx^{n+m}$.

We know, for example, that $2x^5 \times 3x^4 = 6x^9$.
We multiply the coefficients and add the powers and this is shown clearly by the formula above.

EXERCISE 2

If $\frac{a}{b}$ and $\frac{c}{d}$ are two fractions write down a formula for:

a subtracting, **b** multiplying, **c** dividing them.

If $\frac{a}{d}$ and $\frac{c}{d}$ are two fractions (i.e. they have the same denominator) find a formula for:

d adding, **e** multiplying them.

Find **f** $ax^n \div bx^m$, **g** $ax^2 + bx^2$.

Copy and complete the following:

h $a(bx + c) = abx +$

i $ax + ay - az = a(x +$

j $ax : ay = x :$

k $(a + b)^2 = a^2 +$

l $\frac{ax}{bx} =$

m $3n^2 + 3n = n($

REARRANGING FORMULAE

We are familiar with rearranging an equation in order to find x.
It is sometimes necessary to rearrange a formula so that some other letter is isolated on one side.

EXAMPLE 5

Make **b** the subject of the formula $\mathbf{P = aW + b}$.

"Make **b** the subject" means isolate **b** on its own on the left of the equation.
We apply the Sutra *Transpose and Apply.*
Since **b** has **aW** added to it we can subtract **aW** from both sides of the equation. This gives
$\underline{\mathbf{b = P - aW}}$.

EXAMPLE 6

Make m the subject of $\mathbf{E = mc^2}$.

Here **m** is multiplied by $\mathbf{c^2}$ so we divide both sides by $\mathbf{c^2}$ giving $\underline{\mathbf{m} = \frac{E}{c^2}}$.

EXERCISE 3

Make the letter given after the formula the subject:

a $a + 1 = bc$, a

b $x - 3y = z$, x

c $2x + y = z$, y

d $v = u + at$, u

e $V = IR$, R

f $2M = a + b$, M

g $I = 12f$, f

h $x + ab = y + c$, x

i $\frac{y}{a} = \frac{x}{b}$, y

j $2M = a + b$, b

EXAMPLE 7

Make **X** the subject of **$Y = abX + c$**.

Here the **X** is multiplied by **ab** and then **c** is added.
We first take **c** over to get **$Y - c = abX$**.
Then thinking of **ab** as a unit we divide by **ab** on the left: $\underline{X = \frac{Y-c}{ab}}$.

EXERCISE 4

Make the letter given after the formula the subject:

a $v = u + at$, a

b $3x - y = z$, x

c $abc + d = e$, a

d $ax - pq = z$, x

e $X = pY - q$, Y

f $x = pY - q$, p

g $f = \frac{a+b}{n}$, a

h $D = ad - bc$, a

i $AB + CD = EF$, C

APPLICATIONS

You may have used the squaring method and found the duplexes of a, a+b and b. Or you may have used *Vertically and Cross-wise.*

In any case your answer should have been **$(a + b)^2 = a^2 + 2ab + b^2$**.

To verify that the left and right sides of this equation are equal we could choose values for **a** and **b** and show that this gives the same value on each side.

Suppose a=3 and b=4. Then **$(a + b)^2$** $= (3 + 4)^2 = 7^2 =$ **49**.

And **$a^2 + 2ab + b^2$** $= 3^2 + 2\times3\times4 + 4^2 = 9 + 24 + 16 =$ **49** again.

3 Put a=5 and b=7 in each side of this equation and show that both give the same answer.
4 Put a=1 and b= -3 in each side and show that it still works when negative numbers are used.

- Try to find a pair of values for which the two sides are not equal.

The equation **$(a + b)^2 = a^2 + 2ab + b^2$** is true for all values of a and b.

Now suppose we were to rearrange the equation so that a^2 is the subject:

$$\mathbf{a^2 = (a + b)^2 - 2ab - b^2}$$

We could see this as a formula for squaring a number **a**.

The equation is still true for all values of a and b, so if we wanted to find, say, 57^2 we could choose any convenient value for b.

If we choose $b=3$ then the $(a + b)^2$ would be easy to work out because $a+b$ would be 60.
So to get 57^2 using the formula we put $a=57$ and $b=3$.

This gives $57^2 = (57 + 3)^2 - 2\times57\times3 - 3^2$
$= 3600 \quad - 342 \quad - 9$
$= \underline{3249}$.

❋ Check that this is right by some other method.

5 Find 58^2 using the same formula.

6 Find 69^2 using the formula.

If we wanted, say, 61^2 we could still use this method but we would need to put $a=61$ and $b=\bar{1}$ so that $(a+b)$ would still be a convenient number.

7 Find 61^2 by this method. Check your answer.

8 Find 299^2 using the formula.

$(a + b)(a - b)$

The previous formula was based on the expansion (the expansion is the result of multiplying out the brackets) of $(a + b)(a + b)$.

- Find the expansion of $(a + b)(a - b)$.

Multiplying the brackets by *Vertically and Cross-wise* gives $a^2 - ab + ab - b^2$.

The equation $\mathbf{(a + b)(a - b) = a^2 - b^2}$ is also true for all values of a and b.

Since the right-hand side of this equation is just a rearrangement of the left-hand side the two sides will be equal whatever values of a and b are chosen.

- Check this by substituting values for a and b of your choice.

Again we could rearrange the equation to make a^2 the subject:

$$\mathbf{a^2 = (a + b)(a - b) + b^2}$$

We could use this for squaring a number **a**, say 57 again. Putting b=3 as before:

$57^2 = (57 + 3)(57 - 3) + 3^2$
$= 60 \times 54 + 9$
$= \underline{3249}$.

This is easier than the earlier method and is a very useful way of squaring numbers, especially if they are near a base like 60.

In finding 57^2 we could also have used b=7.

9 Find 57^2 using b=7. Do it in your head if you can.

EXERCISE 5

Use the formula $\mathbf{a^2 = (a + b)(a - b) + b^2}$ to square the following mentally:

a 39 **b** 41 **c** 62 **d** 28 **e** 81 **f** 301

As we will see later we often need to find the difference of two square numbers.

For example we may need to find $33^2 - 31^2$.

And the formula $\mathbf{a^2 = (a + b)(a - b) + b^2}$ shows that since $a^2 - b^2$ is the difference of two squares it can also be found using $(a + b)(a - b)$.

EXAMPLE 8

Find $33^2 - 31^2$.

Here we see that a=33 and b=31 so we substitute these into $(a + b)(a - b)$.

This gives $(33 + 31)(33 - 31) = 64 \times 2 = \underline{128}$.

This is actually a very quick method for subtracting squares of numbers: you simply add and subtract the numbers and multiply the two results together.
If you are not convinced of this find $33^2 - 31^2$ by some other method.

This comes under the Sutra *By Addition and by Subtraction.*

EXAMPLE 9

Find $77^2 - 67^2$.

Adding we get 144, subtracting we get 10.
So $77^2 - 67^2 = 144 \times 10 = 1440$.

EXERCISE 6

Use $\mathbf{a^2 - b^2 = (a + b)(a - b)}$ to find (mentally where possible):

a $46^2 - 44^2$
b $38^2 - 36^2$
c $57^2 - 56^2$
d $37^2 - 34^2$
e $168^2 - 167^2$
f $333^2 - 331^2$
g $7654^2 - 7653^2$
h $87^2 - 84^2$
i $88^2 - 78^2$
j $161^2 - 141^2$
k $53^2 - 7^2$
l $(15\frac{1}{2})^2 - (4\frac{1}{2})^2$
m $20^2 - 3^2$

The last question here shows that this method is not always the best one.

24 Squares, Cubes and roots

$1^2 = 1$
$2^2 = 4$
$3^2 = 9$
$4^2 = 16$

1 Above are the first four square numbers. Copy the list and extend it up to 202. Write the list in two or four columns.

● Draw out a 9-point circle. Find the digit sums of your 20 square numbers and place each square number on the branch where it belongs.

2 What do you notice about the digit sums of square numbers?

● Write out the 20 digit sums in a row: 1, 4, 9, 7, ...
What else do you notice about the digit sums?

You should have found that the digit sums can only be 1, 4, 7 or 9 and that they repeat in a cycle of nine: 1, 4, 9, 7, 7, 9, 4, 1, 9.

3 Can a square number have a digit sum of 6?

Make another row of 20 figures- this time the figures are the last figure in each square number: 1, 4, 9, 6,....

4 There are two things to notice about this list. Study the numbers and write down what you think they are.

5 Can a square number end with a 7?

Square numbers only have digit sums of 1, 4, 7, 9
and they only end in 1, 4, 5, 6, 9, 0.

This means that square numbers cannot have certain digit sums and they cannot end with certain figures.

In the exercise below some numbers cannot be square numbers according to the above results. You have to find out which ones they are.

EXERCISE 1

Which are not square numbers (according to the above results):

a 4539 **b** 5776 **c** 6889 **d** 5271 **e** 104976 **f** 65436 **g** 27478 **h** 75379

If a number has a valid digit sum and a valid last figure that does not mean that it is a square number. The last number in the exercise, 75379, is not a square number even though it has an allowed digit sum of 4 and an allowed last figure of 9.

SQUARE ROOTS OF PERFECT SQUARES

We now look at the reverse process: given a square number how do we find the positive number of which it is the square. We call this process finding the square root. We may, for example, want to find the square root of 6889 which can also be written: $\sqrt{6889}$.

So we may write $\sqrt{25} = 5$ and $\sqrt{1600} = 40$ (because $40^2 = 1600$).

EXAMPLE 1

Find $\sqrt{360000}$.

Since $600^2 = 360000$ the answer is 600.

The important thing to notice here is that when we square a number ending in noughts we get twice as many noughts in the answer.
Therefore **the square root will have half that number of noughts.**

So in the Example above, which has four noughts, there will be two noughts in the answer.
And since we recognise that 36 is 6^2 we find the answer to be 600.
In fact it is common practice to mark off pairs of figures starting at the right:

$$\sqrt{36'00'00} = 600$$

Since the number is split into three groups there will be three figures in the answer and each pair of noughts means one nought in the answer.

EXAMPLE 2

Find $\sqrt{1.44}$.

We are looking for a number which, when squared, becomes 1.44.

Since $12^2 = 144$, and $1.2^2 = 1.44$ the answer must be $\underline{1.2}$.

EXAMPLE 3

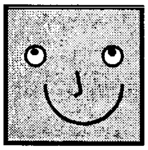
Find $\sqrt{\frac{16}{9}}$.

We take the square root of 16 and 9 and give the answer as $\underline{\frac{4}{3}}$.

EXERCISE 2

Find the square root of the following:

a 9
b 100
c 4900
d 640,000
e 4000,000
f 10,000
g 1.21
h 0.49
i 2.25
j 0.04
k 0.0064
l 0.01
m $\frac{4}{9}$
n $\frac{36}{49}$
o $\frac{2500}{81}$
p $\frac{50}{18}$
q $1\frac{9}{16}$



a perfect square

EXAMPLE 4

Find $\sqrt{6889}$.

First note that there are two groups of figures, 68'89, so we expect a 2-figure answer.

Next we use *The First by the First and the Last by the Last.* Looking at the 68 at the beginning we can see that since 68 is greater than 64 (8^2) and less than 81 (9^2) the first figure must be 8.

Or looking at it another way 6889 is between 6400 and 8100

$$6400 = 80^2$$
$$\mathbf{6889 = 8?^2}$$
$$8100 = 90^2$$

so $\sqrt{6889}$ must be between 80 and 90. I.e.. it must be eighty something.
Now we look at the last figure of 6889, which is 9.
Any number ending with 3 will end with 9 when it is squared so the number we are looking for could be 83.

But any number ending in 7 will also end in 9 when it is squared so the number could also be 87.

So is the answer 83 or 87?
There are two easy ways of deciding. One is to use the digit sums.
If $87^2 = 6889$ then converting to digit sums we get $6^2 = 4$, which is not correct.
But $83^2 = 6889$ becomes $4 = 4$, so the answer must be <u>83</u>.

The other method is to recall that since $85^2 = 7225$ and 6889 is **below** this $\sqrt{6889}$ must be **below 85**. So it must be <u>83</u>.

> To find the square root of a perfect square we find the first figure by looking at the first figures and we find two possible last figures by looking at the last figure. We then decide which is correct either by considering the digit sums or by considering the square of their mean.

EXAMPLE 5

Find $\sqrt{5776}$.

The 57 at the beginning is between 49 and 64, so the first figure must be 7.

The 6 at the end tells us the square root ends in 4 or 6.
So the answer is 74 or 76.

$74^2 = 5776$ becomes $4 = 7$ which is not true in terms of digit sums, so 74 is not the answer.
$76^2 = 5776$ becomes $7 = 7$ so <u>76</u> is the answer.

Alternatively to choose between 74 and 76 we note that $75^2 = 5625$ and 5776 is greater than this so the square root must be greater than 75. So it must be <u>76</u>.

In the following exercise try to find the answers mentally if you can, writing down only the answers.

EXERCISE 3

Find the square root of:

a 2116 **b** 5329 **c** 1444 **d** 6724 **e** 3481 **f** 4489 **g** 8836

h 361 **i** 784 **j** 3721 **k** 2209 **l** 4225 **m** 9604 **n** 5929

As you will have seen, square numbers ending in 5 must have a square root ending in 5- there is only one possibility for the last figure.

EXAMPLE 6

Find $\sqrt{31329}$.

If we mark off pairs of digits from the right here we get 3'13'29.
We have three groups, indicating that the answer is a 3-figure number.

However if you know all the square numbers up to 20^2 we can still get the answer by this method. We think of 31329 as 313'29.

Since 313 lies between 289 (17^2) and 324 (18^2) the first two figures must be 17. And the last figure is 3 or 7, so 173 and 177 are the two possibilities.

The digit sum will then confirm 177 as the right one.
Alternatively you may argue that since 313 is closer to 324 than 289 it will be 177 rather than 173.

EXERCISE 4

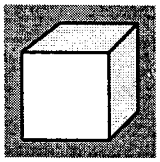
Find the square root of :

a 26896 **b** 32761 **c** 16129 **d** 24964 **e** 36864 **f** 18496

g 21025 **h** 29929 **i** 14161 **j** 11236

CUBING (USING PROPORTIONALETY)

Just as squaring means multiplying a number by itself,
cubing means multiplying a number by itself twice.
So 2 cubed, written 2^3, is 2×2×2 = 8.
And 30^3 = 30×30×30 = 27,000.



6 Make a careful list of all the cube numbers from 1 to 10. Check that all your answers are correct.

These numbers are extremely useful and the following exercise will help you to recognise them.

EXERCISE 5

Which cube number in your list is nearest to:

a 100 **b** 400 **c** 600

Which cube numbers are: **d** also square numbers **e** multiples of 9 **f** odd

Which two cube numbers have: **g** a sum of 35 **h** a sum of 370 **i** a sum of 637

j a difference of 37 **k** a difference of 208

The method of squaring a 2-figure number is illustrated by finding $(a+b)^2$.
We can use *Vertically and Cross-wise* to multiply (a+b) by (a+b).

$$\begin{array}{ccc} a & + & b \\ \underline{a} & \underline{+} & \underline{b} \\ \multicolumn{3}{c}{\underline{a^2 + 2ab + b^2}} \end{array}$$

Vertically on the left we get $a \times a = a^2$, and vertically on the right we get $b \times b = b^2$.
Cross-wise we get ab+ab=2ab.
This verifies the familiar method for squaring 2-figure numbers where we square the left-hand figure, then take twice the product and then square the right-hand figure.

To extend this from squaring to cubing we need to know what we get for $(a+b)^3$.

Since $(a+b)^2 = a^2+2ab+b^2$ we can find $(a+b)^3$ by multiplying $(a^2+2ab+b^2)$ by $(a+b)$:

$$\begin{array}{c} a^2 + 2ab + b^2 \\ \underline{a + b} \\ \underline{a^3 + (a^2b + 2a^2b) + (2ab^2 + ab^2) + b^3} \end{array} = \underline{a^3 + 3a^2b + 3ab^2 + b^3}$$

We have used the moving multiplier method here: vertically on the left, then cross-wise.
Then we move the a+b over to the right and multiply cross-wise and then vertically on the right.

The answer simplifies to the result shown on the right.

And just as $\mathbf{a^2 + 2ab + b^2}$ shows the squaring method $\mathbf{a^3 + 3a^2b + 3ab^2 + b^3}$ shows us how to cube.

$$\mathbf{(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}$$

EXAMPLE 7

Find 41^3.

Here a=4 and b=1 so for 41^3 we find a^3, $3a^2b$, $3ab^2$ and b^3.

We get 64, 48, 12 and 1 and we combine these together in the usual way:

$6\underset{\cup}{4},4\underset{\cup}{8},1\,2,1 = \underline{68921}$ which is the answer.

This method can be simplified further by noting the patterns in the formula for $(a + b)^3$ above.

Note that the power on the letter a goes down by 1 from each term to the next, and that the power on b goes up by 1 as we go from left to right.

This suggests the following method.

EXAMPLE 8

Find 12^3.

We first put down the cube of the first digit: $1^3 = 1$.

Then seeing the ratio 1:2 in the number 12 we extend the ratio until we have four figures in all:

1 2 4 8.

We then double the two middle figures and add them on:

$$\begin{array}{ccccl} \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{8} & \\ & 4 & 8 & & + \\ \hline 1 & 6 & {}_{1}2 & 8 & = \underline{1728} \text{ since the 1 is carried to the 6.} \end{array}$$

EXAMPLE 9

Find 16^3.

We continue the ratio 1:6. **1 6 36 216.**

Then add on double the two middle figures:

$$\begin{array}{ccccl} \mathbf{1} & \mathbf{6} & \mathbf{36} & \mathbf{216} & \\ & 12 & 72 & & + \\ \hline 4 & 0 & 9 & 6 & \end{array}$$

We bring down the 6 at the end on the right.
Then mentally add the 36, 72 and the carried 21 to get 129.
We put down the last figure of this, 9, and carry the 12.
In the next column we add 6, 12 and the carried 12 to get 30, put down the 0 and carry the 3.
This 3 then added to the 1 on the left gives the final 4. So $16^3 = \underline{4096}$.

EXERCISE 6

Cube the following:

a 13 **b** 14 **c** 15 **d** 11

EXAMPLE 10

Cube 31.

We start by cubing 3 and then continue the ratio 3:1.

$$\begin{array}{rrrrl} \mathbf{27} & \mathbf{9} & \mathbf{3} & \mathbf{1} & \\ & 18 & 6 & & + \\ \hline 29 & 7 & 9 & 1 & \\ \hline \end{array}$$

So $31^3 = \underline{29791}$.

EXAMPLE 11

Find 25^3.

Here we start with the cube of 2, which is **8**, then to maintain the ratio we **divide by 2 and multiply by 5** until we arrive at the cube of 5.

8÷2 = 4, 4×5 = **20**,
Then 20÷2 = 10, 10×5 = **50**, and so on:

$$\begin{array}{rrrrl} \mathbf{8} & \mathbf{20} & \mathbf{50} & \mathbf{125} & \\ & 40 & 100 & & + \\ \hline 15 & 6 & 2 & 5 & \\ \hline \end{array}$$

So $25^3 = \underline{15625}$.

EXERCISE 7

Cube the following:

a 21 **b** 41 **c** 51 **d** 23 **e** 24 **f** 22 **g** 42 **h** 32

CUBE ROOTS OF PERFECT CUBES

You will recall looking at the last digit of the first few square numbers and seeing that there are two possibilities for the last digit of the square root of a perfect square.

- Look carefully at the last digits of the first ten cube numbers.

The cubes of 1, 4, 6, 9 end in the same digit (1, 4, 6, 9). It follows that the cube of any number ending in 1, 4, 6 or 9 will end in 1, 4, 6 or 9.

For the other numbers we note: numbers ending in 2, 3, 5, 7, 8
will end in 8, 7, 5, 3, 2 when cubed.
These pairs of numbers all add up to 10.

This means we can tell exactly what the last figure of a cube root is by looking at the last figure of the cube.

To sum up:

> The cube root of numbers ending in 1, 4, 6, 9 will end in 1, 4, 6, 9.
> The cube root of numbers ending in 2, 3, 5, 7, 8 will end in the difference from 10 of the last figure.

EXAMPLE 12

Find the cube root of 12,167.

$$\begin{aligned} 20^3 &= 8{,}000 \\ \mathbf{2?^3} &= \mathbf{12{,}167} \\ 30^3 &= 27{,}000 \end{aligned}$$

$20^3 = 8000$, and $30^3 = 27000$ and 12,167 lies between 8,000 and 27,000.
Therefore the cube root of 12,167 must lie between 20 and 30.
That is, it must be a 2-figure number, starting with 2.

Next note that 12,167 ends in a 7 and looking at your list of cubes you will see that only numbers ending in 3 will end in 7 when cubed.
So the last figure is clearly 3 and the answer is <u>23</u>.

So cube roots of perfect cubes are particularly easy because the first and last figures are easily found from the first and last figures of the cube.

EXERCISE 8

Find the cube root of: **a** 39,304 **b** 238,328 **c** 456,533

d 205,379 **e** 17,576 **f** 804,357 **g** 110,592 **h** 91,125

25 Equations

You are familiar with the solution of equations like $2x - 5 = 7$ and $\frac{3x}{5} + 4 = 1$.
In this chapter we extend the range of equations we can handle to those with fractional answers and other types of equation. We begin with some revision.

EXERCISE 1

Solve the following mentally:

a $2x - 5 = 7$ **b** $3x + 4 = 25$ **c** $2x + 13 = 5$ **d** $5x - 1 = \overline{16}$

e $\frac{x}{3} - 3 = 4$ **f** $\frac{2x}{3} + 4 = 10$ **g** $\frac{x}{2} - 3 = \bar{7}$ **h** $\frac{3x}{4} + 4 = 1$

SOME VARIATIONS

Equations may not always be in the standard forms above.

EXAMPLE 1

Solve $23 = 2 + 3x$.

This is just the same as $3x + 2 = 23$ and so we mentally take the 2 from the 23 and divide by 3. We get $x=7$.

EXAMPLE 2

Solve $2x + 13 + x = 4$.

Here we mentally simplify the left-hand side of the equation by combining the 2x and the x to mentally get $3x + 13 = 4$, so that $x = \bar{3}$.

EXAMPLE 3

Solve $16 = 5 - 2x$.

Here we see -2x on the right-hand side of the equation.
We mentally put this on the left-hand side because the minus then becomes a plus, and take the 16 to the right to get $2x = 5 - 16$.

- Make sure you understand what has happened here: the -2x is transposed to 2x on the other side, and the 16 (which is +16) is transposed to -16 on the other side. The 5 is unaltered.

This gives $x = -5\frac{1}{2}$.

If you like you can take the 5 in 16 = 5 - 2x to the left to get 16 - 5 = -2x which becomes 11 = -2x. Then dividing 11 by -2 we get $x = -5\frac{1}{2}$ as before.

EXERCISE 2

Solve the following:

a $3 + 4x = 23$
b $17 = 1 + 2x$
c $2x + 3x = 25$
d $x + 1 + x = 11$
e $x - 3 + 3x = 13$
f $30 = 2x + 3x$
g $7 + x = 50 - 3$
h $3x + 2 + 2x - x = 3$
i $5 = 1 - x$
j $18 = 22 - 2x$
k $16 = 1 - 5x$
l $7 - 3x = 19$

FRACTIONAL ANSWERS

EXAMPLE 4

Solve $3x - 5 = 8$.

Using *Transpose and Apply* we first get $3x = 13$,
then since 3 does not divide exactly into 13 we write $x = \frac{13}{3}$.

$x = 4\frac{1}{3}$ is also correct but we will leave answers in top-heavy form for the rest of this chapter.

EXAMPLE 5

Solve $18x = 2$.

The answer is simply $x = \frac{2}{18}$ which cancels down to $x = \frac{1}{9}$.

EXAMPLE 6

Solve $\frac{x}{3} = \frac{4}{5}$.

Since we need x here we transpose the 3: $x = 3\times\frac{4}{5}$.
So $\underline{x = \frac{12}{5}}$.

It is sometimes useful to cross-multiply in equations like the one in Example 6.
This means we multiply diagonally and equate the two products: $x\times5 = 4\times3$, which gives $5x = 12$ and $x = \frac{12}{5}$ again.
This is particularly useful with equations like the one below.

EXAMPLE 7

Solve $\frac{5}{x} = \frac{2}{3}$.

Cross-multiplying gives $2x = 15$ and $\underline{x = \frac{15}{2}}$.

EXERCISE 3

Solve the following (cancel but leave as top-heavy fractions where appropriate):

a $3x = 5$ **b** $8x = 1$ **c** $3x - 4 = 7$ **d** $5x + 4 = 5$ **e** $3 + 2x = 10$

f $9 = 1 + 5x$ **g** $3x + 12 = 4$ **h** $2 - 3x = 6$ **i** $4x + 3 = \bar{6}$ **j** $2 - 4x = \bar{5}$
k $\frac{3x}{2} + 1 = 5$ **l** $\frac{4x}{5} - 2 = 4$ **m** $\frac{x}{5} = \frac{8}{3}$ **n** $\frac{x}{6} = \frac{3g}{4}$ **o** $\frac{2x}{3} = \frac{1}{4}$

p $\frac{2}{1} = \frac{3x}{5}$ **q** $\frac{3}{x} = \frac{2}{7}$ **r** $\frac{2}{3x} = \frac{6}{5}$ **s** $\frac{1}{3} = \frac{4}{3x}$ **t** $\frac{1}{5} = \frac{1}{3x}$

TWO x TERMS

So far none of our equations have had an x term on both sides of the equation.

EXAMPLE 8

Solve $5x + 3 = 3x + 17$.

The method here is to collect all the x terms on one side of the equation and all the other terms on the other side.

In the above equation we take the $3x$ to the left side and the 3 to the right.
This gives $5x - 3x = 17 - 3$.
So $2x = 14$,
and $\underline{x = 7}$.

Note carefully how the terms which have changed side have also changed sign. The other terms are left unchanged.

EXERCISE 4

Solve the following:

a $7x + 1 = 5x + 11$
b $8x + 3 = 3x + 23$
c $4x + 7 = x + 16$
d $5x + 1 = 2x + 19$
e $2x + 3 = x + 1$
f $10x + 17 = 5x + 2$

These equations can also be solved mentally. We can see how many x's there will be on the left and what the number on the right will be when we have transposed. Then we just divide the number on the right by the number on the left.

So for example in question (a) above we see there will be $2x$ on the left when the $5x$ is taken over, and that there will be 10 on the right when the 1 is taken over.
Then we just divide 10 by 2 to get $x=5$.

In the next exercise write down only the answer.

EXERCISE 5

Solve the following mentally:

a $5x + 3 = 3x + 15$
b $3x + 1 = x + 21$
c $7x + 5 = 4x + 20$
d $8x + 3 = 3x + 28$
e $10x + 11 = 3x + 32$
f $9x + 17 = x + 81$

EXAMPLE 9

Solve $7 - 2x = x - 5$.

Here seeing the $-2x$ it is best to collect the x terms on the right.
Then the $-2x$ will become $+2x$ on the right, and the -5 will be $+5$ on the left.
This gives $12 = 3x$ (mentally).
So $\underline{x = 4}$.

EXAMPLE 10

Solve $2(3x + 4) = 2x + 20$.

Mentally we see there will be $4x$ on the left and 12 on the right.
So $\underline{x = 3}$.

EXAMPLE 11

$6x + 5 = 4x - 21$.

There will be $2x$ on the left and -26 on the right (since $-21-5 = -26$).
So $\underline{x = -13}$.

EXERCISE 6

Solve the following mentally:

a $7x - 5 = 4x + 10$

b $5 + 4x = 13 + 2x$

c $7x + 3 = 15 + x$

d $5x - 21 = x - 1$

e $6x + 1 = 4x - 3$

f $5x - 1 = 3x + 9$

g $8x - 9 = 5x + 12$

h $8x - 2 = 6x - 23$

i $10x + 1 = 25 - 2x$

j $2(3x + 1) = x + 27$

k $3(x - 3) = 2x + 8$

l $14 - 3x = x + 10$

m $17 - 5x = 8 - 2x$

n $9x - 3 = -x + 57$

o $4x - 7 = x + 9$

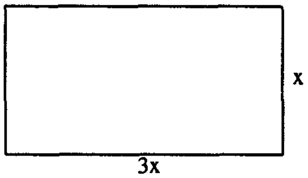
FORMING EQUATIONS

Equations are very useful in solving problems when the problem can be put into mathematical form.

EXAMPLE 12

A rectangle is 3 times as long as it is wide. If its perimeter is 52cm find its length and width.

The fact that the rectangle is 3 times as long as it is wide can be put into mathematical form by calling the shorter side **x**, because we can then call the longer side **3x**.



Since the perimeter is given as **52** we can write the equation:

$$\mathbf{3x + x + 3x + x = 52.}$$

Which means $8x = 52$ and $x = 6½$.

So the answer to the question is 19½ and 6½.

Putting a problem into mathematical form like this is often very useful and the first step is to decide what can usefully be called x. We then interpret the rest of the given information in terms of x, as shown above.

EXAMPLE 13

Three consecutive numbers add up to 114. What are they?

If we call the first number **n**, then **n+1** must be the next number (since they are consecutive) and **n+2** is the third. It is usual to use n rather than x if it is known to be a whole number.

We are told that they add up to **114** so: $\mathbf{n + n + 1 + n + 2 = 114}$.
Therefore $3n + 3 = 114$ and $n = 37$.

So the numbers are <u>37, 38 and 39</u>.

EXAMPLE 14

Find two consecutive odd numbers such that the sum of 5 times the smaller and twice the greater is 95.

We have seen that $\mathbf{2n + 1}$ represents an odd number so the next odd number will be $\mathbf{2n + 3}$.

So the question says that $\mathbf{5(2n + 1) + 2(2n + 3) = 95}$.
Multiplying out the brackets gives $10n + 5 + 4n + 6 = 95$.
So $14n + 11 = 95$ and $n = 6$.
Putting this value into $2n + 1$ and $2n + 3$ gives <u>13 and 15</u>.

EXERCISE 7

Express the following as an equation and find the number(s):

a When 30 is added to 3 times a number the result is 66.

b A rectangle is twice as long as it is wide and its perimeter is 66.

c 4 consecutive numbers add up to 142.

d Three consecutive even numbers add up to 72 (call the first even number 2n).

e 2 consecutive odd numbers add up to 104.

f There are two consecutive numbers and the greater added to 3 times the smaller makes 53.

g A room is 3m longer than its width and its perimeter is 38m.

QUADRATIC EQUATIONS

EXAMPLE 15

Solve the equation $x^2 = 25$.

This says that a number squared comes to 25.
The number is clearly 5 as $5^2 = 25$. So $x = 5$.

However this is not the whole answer because $(-5)^2 = 25$ as well.
So the full answer is $\underline{x = 5 \text{ or } x = -5}$.

This can also be written more briefly as $\underline{x = \pm 5}$ which means exactly the same.

EXAMPLE 16

Solve $3x^2 = 48$.

We divide both sides by 3 to get $x^2 = 16$.
So $x = \pm 4$.

EXAMPLE 17

Solve $x^2 + 3 = 15$.

Here we take 3 from both sides to get $x^2 = 12$.
But in this case 12 is not a square number so we write $\underline{x = \pm\sqrt{12}}$.

EXAMPLE 18

Solve $2x^2 - 100 = 1700$.

Here we have to add 100, divide by 2 and take the square root.
This can all be done mentally to get $\underline{x = \pm 30}$.

Equations involving x^2, like this, are called **quadratic equations.**

EXERCISE 8

Solve the following:

a $x^2 = 1600$

b $3x^2 = 300$

c $x^2 + 5 = 69$

d $x^2 - 600 = 3,000$

e $5x^2 = 45$

f $4x^2 - 4 = 200$

g $9x^2 - 9 = 0$

h $2x^2 + 3 = 53$

i $4x^2 - 7 = 2$

j $x^2 - 3 = 1222$

k $2x^2 = 882$

l $\frac{x^2}{3} = 27$

"All things began in order, so shall they end, and so shall they begin again; according to the ordainer of order and mystical mathematics of the city of heaven. "

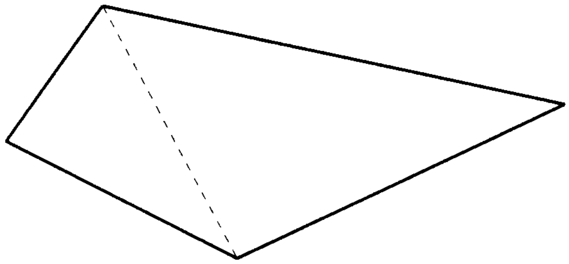
- Sir Thomas Browne (1605-82)

26 Polygons

The word polygon means many-sided figure.
We have met these before: triangles, quadrilaterals, pentagons etc. are all polygons.

ANGLE SUM OF POLYGONS

We know that the angles in any triangle add up to 180°, no matter what shape the triangle has.
We now look at the sum of angles of other polygons.



- Copy the quadrilateral shown above. It does not have to be exactly the same, as long as it is a quadrilateral. Draw the dashed line as well.

The dashed line splits the shape into two triangles.

- Draw arcs to show the three angles of the triangles on the left.

1 What do these angles add up to?

- Draw arcs to indicate the angles of the right-hand triangle.

2 What do these angles add up to?

- With a colour mark the four angles of your quadrilateral.

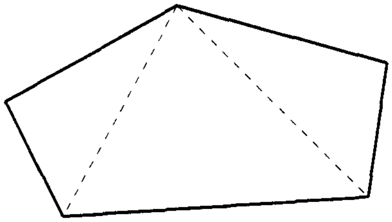
You should now find that the four angles of the quadrilateral are the same as the six angles of the two triangles.

3 Since the angles of each of the triangles add up to 180° the angles of the two triangles together must be twice 180°. What do you think the angles of the quadrilateral add up to?

4 Do you think you would have got the same answer if the quadrilateral had been a different shape of quadrilateral?

5 Copy and complete: "The angles of every quadrilateral add up to. . ."

We now look at pentagons.



Here the pentagon has been divided into three triangles by the two dashed lines.

● Draw a pentagon of your own and divide it into three triangles. In each triangle mark the three angles as before and with a colour mark the five angles of the pentagon.

The total of the nine angles of the three triangles is the same as the five angles of the pentagon.

6 Check that you agree with this and write down what you think the angles of the pentagon add up to.

7 Now draw a hexagon and split it into triangles by drawing diagonal lines from one vertex. In the same way determine the total of the angles inside the hexagon.

8 Copy the table below and continue with more polygons until you can complete the table. Do not try to draw the 20-sided figure: you should be able to tell how many triangles there will be in it.

Number of sides	3	4	5	6	7	10	20
Total of angles	180×1= 180	180×2 = 360	180×3=				

The angles inside a polygon are called the interior angles.

9 What is the connection between the number of sides a polygon has and the number of triangles it can be divided into?

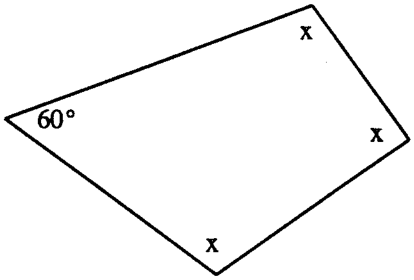
10 If a polygon has n sides how many triangles will it have?

11 What is the connection between the number of triangles a polygon is divided into and the total of its interior angles?

12 If a polygon has n sides what will be the sum of the interior angles? (You may need to study the table you have completed to answer this question)

EXAMPLE 1

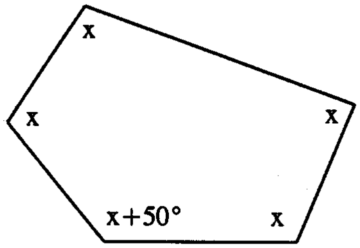
Find x.



Since the total of the angles is 360° we get $3x + 60 = 360$.
The solution to this is $\underline{x = 100°}$.

EXAMPLE 2

Find x in the figure below.



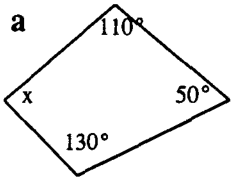
The figure is clearly a pentagon
and the interior angles of all pentagons add up to 540°.

$$
\begin{aligned}
\text{So } x + x + x + x + (x + 50) &= 540, \\
5x + 50 &= 540, \\
x &= 98°.
\end{aligned}
$$

EXERCISE 1

Find x:

a



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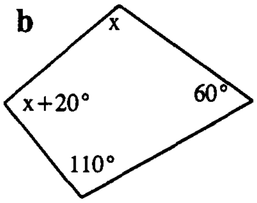
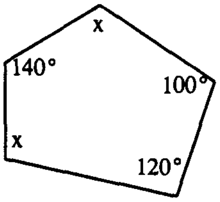
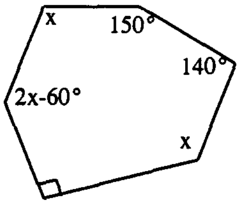


c



d



e A dodecagon (12 sides) has 6 interior angles equal to each other and the other 6 equal to 160°. What is the size of the unknown angles?

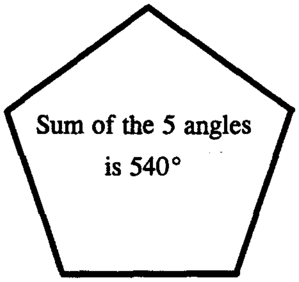
REGULAR POLYGONS

A regular polygon, you may recall, has all its sides and all its interior angles equal. So an equilateral triangle is a regular triangle and a square is a regular quadrilateral.

Since all the angles in a regular polygon are equal we can easily find what each angle is by dividing the total of all the interior angles by the number of sides.

EXAMPLE 3

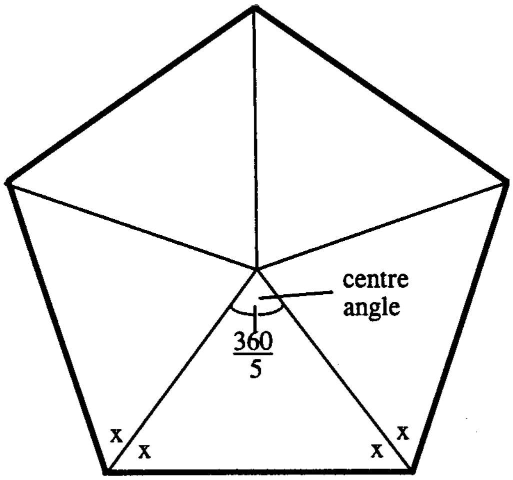
Find the interior angle of a regular pentagon.



We know that the sum of the interior angles is 540° from the earlier work.
And since there are 5 sides and 5 angles in a pentagon we can just divide 540 by 5.
So the interior angles of a regular pentagon are 108°.

Alternative Method

Another way of finding the interior angles of a regular polygon is to first join the centre of the polygon to each of the vertices. And then find the centre angle by dividing a full circle (360°) by the number of sides the polygon has (in this case 5):



This gives 72° for the centre angle. Then looking at the isosceles triangle at the bottom of the figure above we can take 72 from 180 to get 2x = 108°. But since each interior angle is 2x (see the figure above) this is the answer: the interior angles are <u>108°</u>.

> We can get the interior angle of a regular polygon
> by dividing 360 by the number of sides
> and taking the result from 180.

One advantage of this second method is that it can be reversed to give the number of sides when the interior angle is known.

EXAMPLE 4

Find the number of sides of a polygon with interior angles of 108°.

We get the centre angle by taking 108 from 180 to get 72.
Then we divide 72 into 360 to get <u>5 sides</u>.

EXERCISE 2

Using either or both of the above methods find the interior angles of the following regular polygons:

a hexagon **b** octagon **c** nonagon (9 sides) **d** decagon **e** dodecagon **f** 20-sided polygon

Find the number of sides of a regular polygon with interior angles of:

g 156° **h** 160° **i** 168°

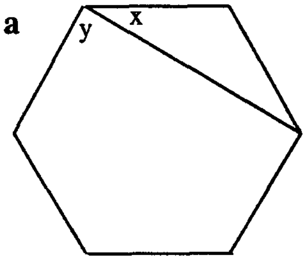
OTHER POLYGON ANGLES

The following example illustrates two methods of finding other angles inside regular polygons: one using isosceles triangles and the other using the centre angle.

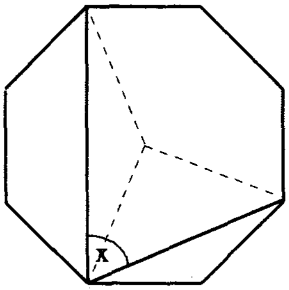
EXAMPLE 5

Find x and y in the following regular polygons:

a



b



a Since the polygon is regular the triangle above is isosceles and since the angles in a regular hexagon are 120°, x must be 30°.
y is easily found because x+y is 120° (another interior angle). So y = 90°.

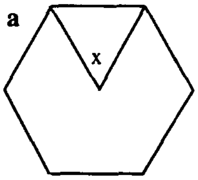
b The centre for a regular octagon is $\frac{360}{8} = 45°$.
The angle x required is the sum of two angles and each part can be found from each of the isosceles shown dashed. The left triangle has an angle of 135° (3 × 45°) at the centre so that the first part of x is 22½°.
The right-hand triangle has an angle of 90° (2 × 45°) at the centre so that the other part of x is 45°.

So x = 22½ + 45 = 67½°.

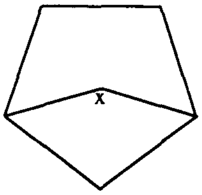
EXERCISE 3

Find x the following regular polygons:

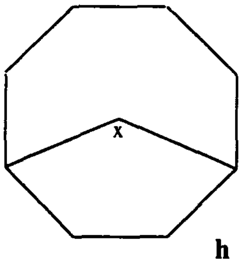
a



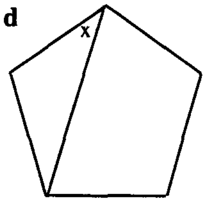
b



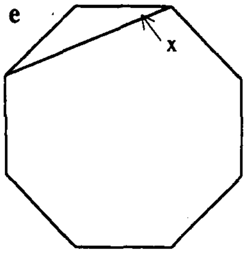
c



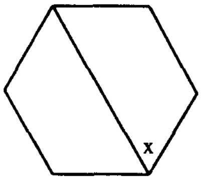
d



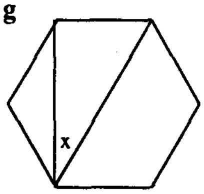
e



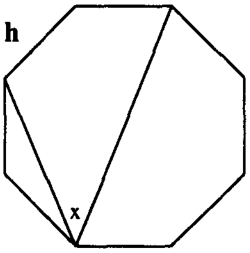
f



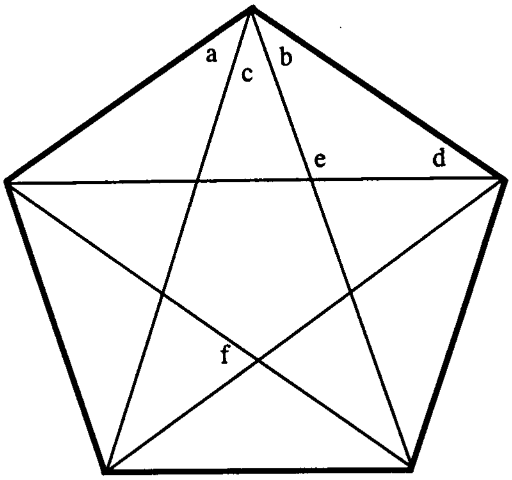
g



h



i Find the angles a to f in the pentagram below:

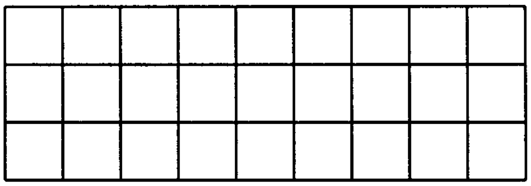


TESSELLATIONS

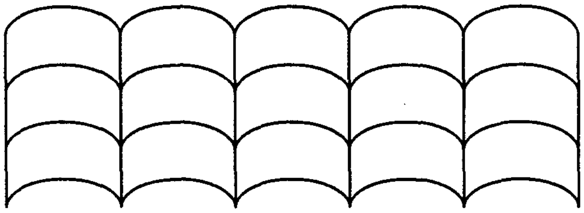
If we need to tile a floor or construct a mosaic we need to ensure that there are no gaps left between the shapes when they are put together.

If certain shapes can be put together without gaps we say that they tessellate.
And the shape they make is called a tessellation.

Perhaps the most obvious tessellation is the one using squares. We can tile a floor if all the tiles are squares of the same size.



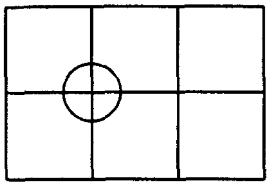
One variation might be to stretch the squares:



Another might be to make some of the sides curved:

But we will just consider the possibilities when all the tiles are regular polygons.

If you look at the tessellation of squares you will see that there are no gaps left because four squares fit exactly around a point:



This is because the 90° angle in the square divides exactly into 360°.

13 Two other regular polygons have an interior angle which divides exactly into 360°. What are they? Draw diagrams to show how they tessellate.

14 Copy and complete the following: "There are just 3 plane tessellations using only one shape of regular polygon. These are the square, the .... and the ....".

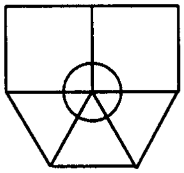
SEMI-REGULAR TESSELLATIONS

Only three regular polygons will tessellate if used on their own.

Things get a lot more interesting when we allow use of two or more regular polygons in our tessellations.

Using the same method as above for regular tessellations we can investigate these tessellations by considering a selection of polygons so that we get 360° by adding one interior angle of each polygon. This ensures that they will fit around a point without leaving any gaps or having any overlaps.

For example, if we take 3 triangles and 2 squares these can be fitted together because adding one interior angle of each of these polygons gives 60° + 60° + 60° + 90° + 90° = 360°:



Although these five shapes will fit together around a point it does not follow that it is a tessellation. We need to be able to continue to add polygons to the shape so that there are 3 triangles and 2 squares at every vertex.

If this is possible and the order of the polygons is the same at each vertex (that is, 2 squares followed by 3 triangles) then the tessellation is called a semi-regular tessellation.

15 Draw the arrangement out carefully and add squares and triangles so that there are 2 squares and 3 triangles, in that order, at every vertex (you may find Worksheet 5 useful for this).

A shorthand for this semi-regular tessellation is 3, 3, 3, 4, 4 (or 4, 4, 3, 3, 3 or 3, 4, 4, 3, 3 etc.). This indicates that there are 3 triangles and 2 squares at every vertex).

Now consider the arrangement 3, 3, 4, 3, 4.

There are still 3 triangles and 2 squares, so these will fit together around a vertex. But the order around the vertex has to be triangle, triangle, square, triangle, square.

16 Draw the arrangement and extend the pattern so that there are 3 triangles and 2 squares at every vertex, and in the order given above.

If certain regular polygons fit around a point it does not follow that they will tessellate.

17 Draw the arrangement 5, 5,10 using Worksheet 5 and try to add more polygons to see if this arrangement will tessellate. You should find that it cannot be done.

EXERCISE 4

For each of the following draw the basic arrangement and extend the pattern until you are convinced it is semi-regular or not. Two are semi-regular and one is not (use triangular spotty paper):

a 3, 6, 3, 6 **b** 3, 3, 3, 3, 6 **c** 3, 3, 6, 6

There exactly eight semi-regular tessellations. Four of them you have met above and the other four are included in the next exercise.

To summarise:

There 8 semi-regular tessellations.

Semi-regular tessellations must:

- **a** fit round a point without leaving gaps or overlaps,
- **b** be extendible,
- **c** have the same order of polygons at each vertex.

EXERCISE 5

The arrangements below include 4 semi-regular tessellations and 3 others.
Of the 3 others one will not fit around a point and two cannot be extended.
Find the 4 semi-regular tessellations:

a 4, 6, 12

b 4, 8, 8

c 3, 9, 18

d 3, 4, 6, 4

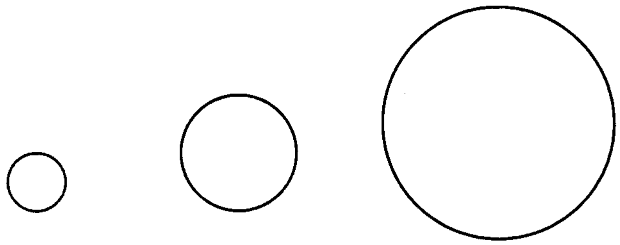
e 3, 4, 3, 12

f 3, 12, 12

g 3, 4, 5, 5

18 Make a complete list of all 8 of the semi-regular tessellations.

27 Similar Figures



We frequently see objects which are the same shape but different in size.
A boy and a man are about the same shape, but their sizes are different.

Figures which have the same shape are called similar figures.

● Name a pair of objects which are similar (or approximately similar).

Since similar figures have the same shape it follows that any angles in one figure must be equal to the corresponding angles in the other figure.

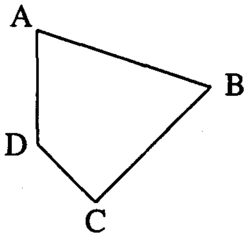


Fig. 1

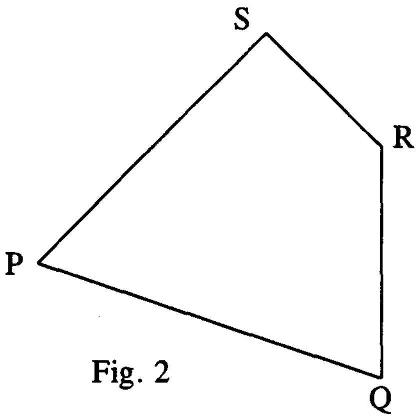


Fig. 2

1 Do you think the figures above are similar? Measure the angles with a protractor to find out.

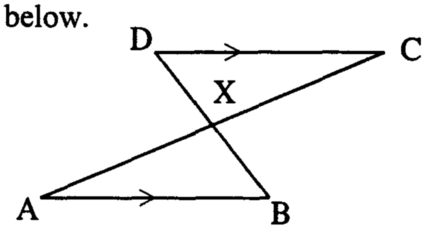
2 Which angle in Fig. 2 corresponds to angle A?
Which angles correspond to B, C and D?

If we want to say that two shapes are similar by referring to the letters at the vertices of the shapes it is usual to order the letters so that they correspond: Since A corresponds to Q, B to P, C to S and D to R we would say ABCD is similar to QPSR.
Labelled in this way if we were told that the two shapes were similar we would know which angles corresponded.

In the following work we will use the symbol 'Δ' for 'triangle'.

EXAMPLE 1

Describe a triangle similar to ΔABX below.



Because there is a pair of parallel lines the angles at A and C are equal (they are Z-angles). Similarly the angles at B and D are equal.
The angles at X, in each triangle, are equal (they are vertically opposite).

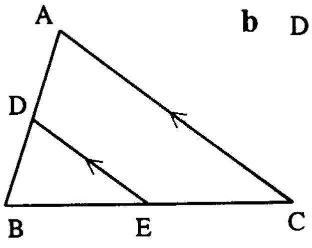
The triangles are therefore similar and we can say ΔABX is similar to ΔCDX.

Note the order of the letters is carefully chosen: if we write ABX we must write CDX because A=C, B=D and X=X.

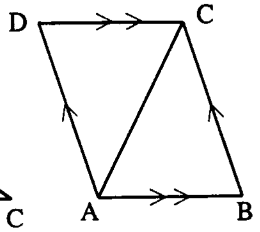
EXERCISE 1

In each of the four questions below give a triangle similar to DABC (do not measure the triangles as they are not all drawn accurately):

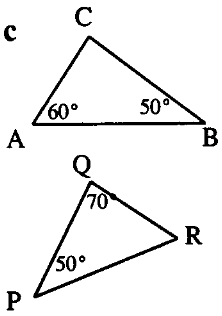
a



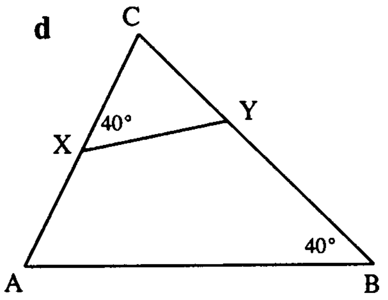
b



c

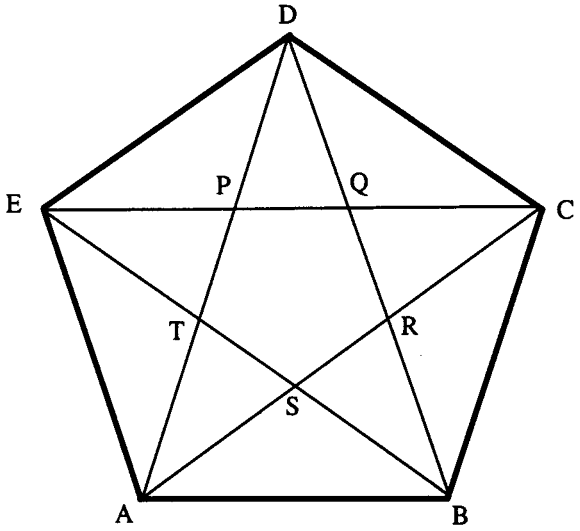


d



e In the pentagram below write down a triangle similar to DECD which has DC as one of its sides.

f Also in the pentagram write down two triangles similar to OPQD which have DC as one of their sides.



3 Measure the lengths AB, BC, CD and DA in Fig. 1 to the nearest mm and insert them in a copy of the table below. Also measure the corresponding lengths in Fig. 2 and insert them in the table.

	AB	BC	CD	DA
Length in Fig. 1				
Corresponding length in Fig. 2				

4 What do you notice about the two rows of numbers?

Fig. 2 is clearly an enlargement of Fig. 1 and the scale factor of the enlargement is 2. Or we could say that Fig. 1 is an enlargement of Fig. 2 with a scale factor of 1/2.

So the scale factor can be found by measuring two corresponding sides and dividing.

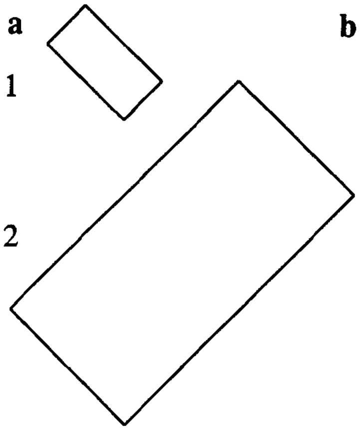
EXERCISE 2

In the four questions below decide, by measurement, if the figures are similar or not, and if they are, give the scale factor of the enlargement from Fig. 1 to Fig. 2 (one pair are not similar):

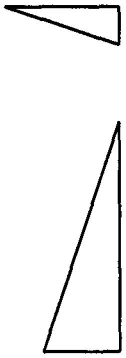
Fig. 1

Fig. 2

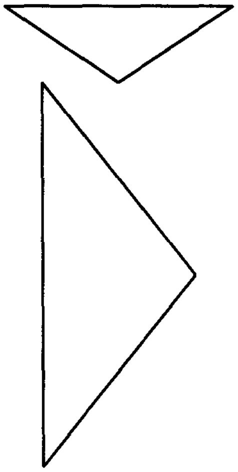
a



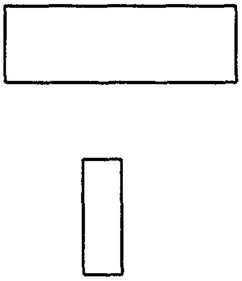
b



c

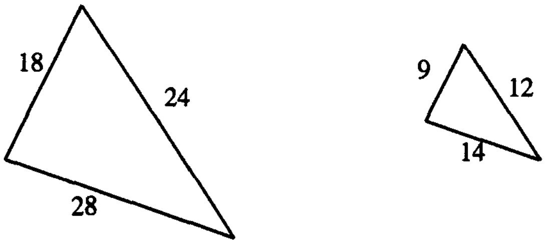


d



EXAMPLE 2

Are these triangles similar?



The ratios of corresponding sides are 28:14, 18:9, 24:12. And these ratios are all equal. The triangles are similar.

5 How do you know that these ratios are all equal?

EXERCISE 3

The five questions below list the lengths of the sides of five pairs of triangles.
You have to decide if each pair are similar, and if they are to give the scale factor of the enlargement (one pair are not similar):

a 2, 5, 4 and 4, 10, 8

b 18, 15, 12 and 6, 5, 4

c 7, 7, 8 and 14, 16, 14

d 20, 31, 42 and 62, 60, 84

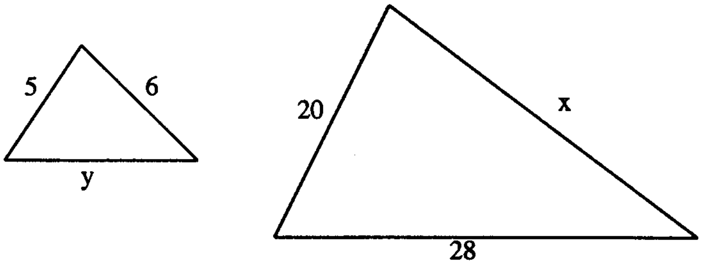
e 30, 40, 50 and 45, 60, 75

If two figures are similar the ratio of every pair of corresponding sides is the same.

So, in the case of Figs. 1 and 2 at the beginning of this chapter, the ratio of any side in Fig. 2 to the corresponding side in Fig. 1 is 2:1.

EXAMPLE 3

Given that the triangles below are similar, find x and y:



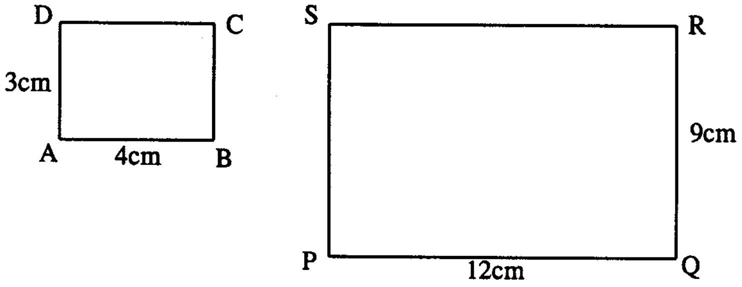
not to scale

Looking at the corresponding sides of length 5 and 20 we see that the scale factor is 4. The larger triangle is 4 times larger than the smaller triangle.

This means that all sides in the larger triangle are 4 times those in the smaller triangle. The value of x must therefore be 24, $\underline{x = 24}$.
And y must be ¼ of 28, so $\underline{y = 7}$.

EXAMPLE 4

Given AC = 5cm find PR.

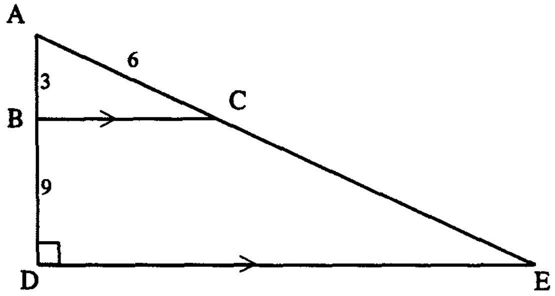


Here we are not told that the figures are similar. But since the base and height of the larger rectangle are 3 times those of the smaller one they must be similar.

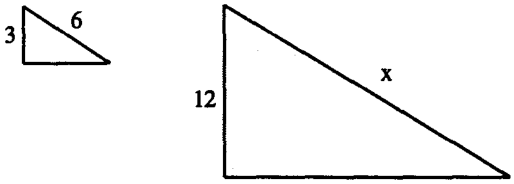
It follows that all lengths, including the diagonals, are 3 times bigger in the larger rectangle. And since AC = 5 then PR must be 15: $\underline{PR = 15cm}$.

EXAMPLE 5

Find AE in the figure below.



The parallel lines show us that ΔABC is right-angled at B.
So triangles ABC and ADE must be similar as they have angle A in common.
It is sometimes useful to draw the triangles out separately when they overlap like this:

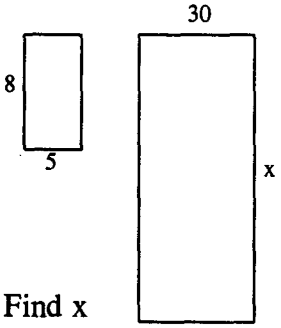


Now we can see that since 12 is 4 times 3, x must be 4 times 6.
So x = 24.

EXERCISE 4

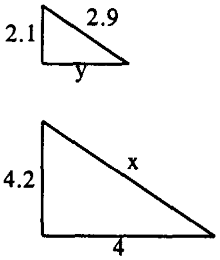
In the first three questions below assume the figures are similar.
Do them mentally as far as possible:

a



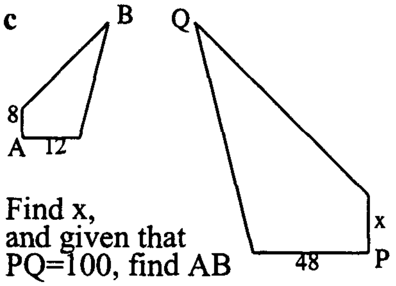
Find x

b



Find x and y

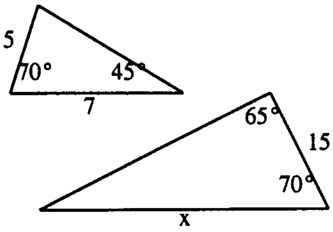
c



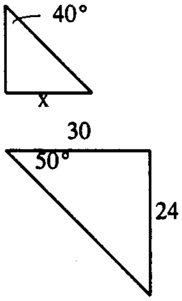
Find x,
and given that
PQ=100, find AB

In the three questions below you will need to check that the triangles are similar to see which sides correspond. Find x in each case: ☞

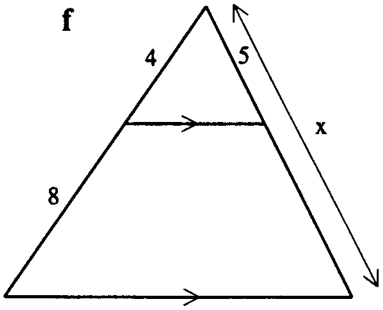
d



e

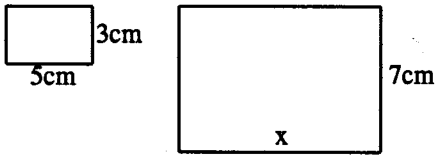


f



EXAMPLE 6

The dimensions of a rectangle are 3cm by 5cm. A similar rectangle has a shortest side of 7cm. Find, to 2 decimal places, the length of the longer side.



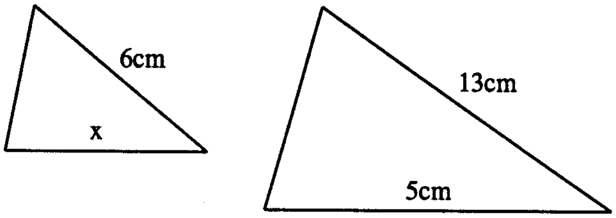
Since 7 is not a multiple of 3 we can tackle this problem as follows.
The ratio of corresponding sides is the same and so x:5 = 7:3.
This can be expressed more usefully by writing the ratios as fractions:

$\frac{x}{5} = \frac{7}{3}$ so that $x = \frac{35}{3} = 11\frac{2}{3}$.

And since we know $\frac{2}{3} = 0.\dot{6}$, $x = 11.66\dot{6}$ = <u>11.67cm</u> to 2 decimal places.

EXAMPLE 7

Find x to 2 decimal places in the similar triangles below:



In terms of ratios we have x:5 = 6:13.
And as fractions: $\frac{x}{5} = \frac{6}{13}$.
This gives $x = \frac{30}{13} = 2\frac{4}{13}$.

We can convert this to a decimal using recurring decimals: we write $\frac{4}{13}$ as $\frac{12}{39}$ and proceed by dividing 12 by the Ekadhika, 4:

So $x = 2\frac{12}{39} = 2.3_307\ldots$ = <u>2.31cm</u> to 2 decimal places.

EXAMPLE 8

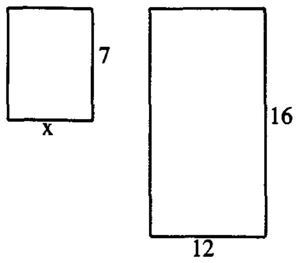
Two model cars are similar in shape. If their lengths are in the ratio 3:5 and the smaller car has a height of 2.4cm find the height of the larger car.

Here if we call the required height x we can write $\frac{x}{2.4} = \frac{5}{3}$. So $x = \frac{12}{3} = 4$cm.

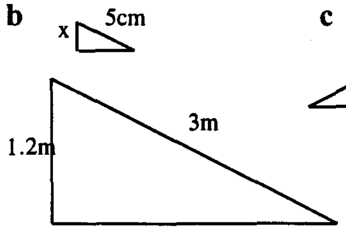
EXERCISE 5

Find x:

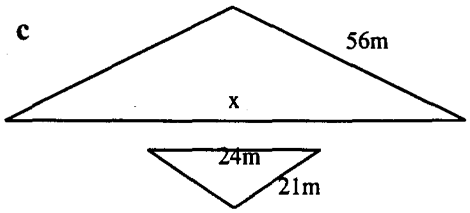
a



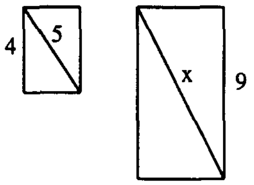
b



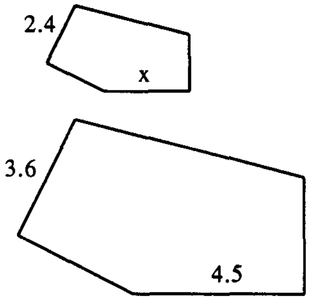
c



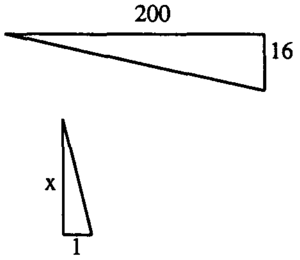
d



e



f



g Two similar rectangles have lengths in the ratio 4:7. If the longer side of the larger rectangle has a length of 3cm what will be the longer side of the smaller rectangle?
Give your answer **i** exactly, **ii** to 2 d.p.

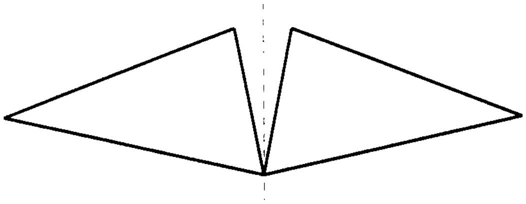
h A photograph measures 3cm by 8cm. An enlargement of the photograph is to have a longest side of 15cm. What will be the length of the shorter side? Give the exact answer as a decimal.

i A man has a height of 1.7m and his son has a height of 1m. Assuming they are similar figures find the man's hand span given that the boy's hand span is 13cm. Give the exact decimal answer.

CONGRUENT FIGURES

Congruent figures are the same shape and the same size.

If a triangle is reflected in a line the reflected triangle is congruent to the original one.



EXERCISE 6

a A quadrilateral is rotated 90° about one corner. Will the two quadrilaterals be congruent?

b A rectangle is cut along a diagonal. Are the two parts congruent?

c Is a circle of radius 5 cm congruent to a circle of diameter 10 cm?

d A shape A is transformed to B. Is B congruent to A if the transformation is a
- **i** translation,
- **ii** rotation,
- **iii** reflection,
- **iv** enlargement?

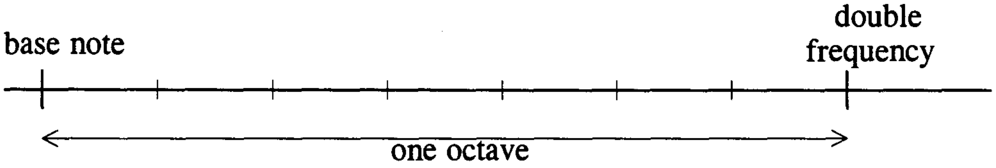
28 The Musical Scale

Sound is a vibration- musical sounds or notes are smooth and some notes sound higher or lower than others.

Musical notes have a definite frequency of vibration which is the number of vibrations that occur in one second. If the frequency is high the note sounds high and if the frequency is low the note sounds low.
So a note of frequency 440 vibrations per second is higher than one of 264 vibrations per second.

If a note of a particular frequency is chosen as a starting point then other notes can be related to it. Suppose we start with a frequency of 264 vibrations per second. Then a note of frequency 528 has twice the frequency and, though higher, has a similar sound.

All musical scales are based on a starting note and the note which has double the frequency of the starting note.



This is called the octave. Other notes are placed inbetween these two notes.

In the usual system there are eight notes including the lowest and highest notes.

But there have been many other systems for dividing the octave in the past and still today different systems are used.

The number of notes in the octave depends on what system is used and even if the number of notes chosen is eight there are different ways of selecting them.

THE NOTES OF THE OCTAVE

The modern, western scale divides the octave into eight notes which means that six notes are to be inserted between the top and bottom notes of the octave.

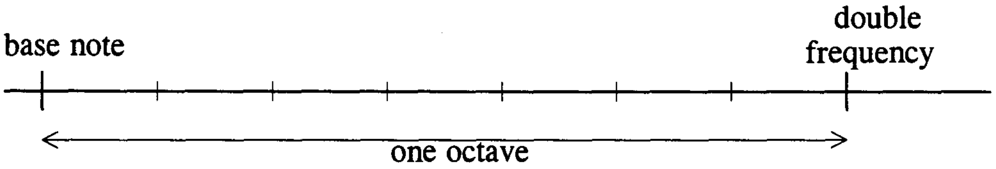
How are these six notes to be determined?
In fact it has been found that when the frequencies of two notes are in the ratio of small numbers the two notes have a pleasing sound in relation to each other.

If the lower and upper notes of the octave are 264 and 528 the ratio is 1:2.

We will shortly see that the other notes of the octave have simple ratios to the base note.

First we will consider the ratio of each note to its preceding note.

The names of the eight notes of the octave are:



EXAMPLE 1

Given the frequency of the base note (**Do**) as 264 and that the ratio of **Do** to **Re** is $\frac{9}{8}$ find the frequency of **Re**.

This simply means that the frequency of **Re** is nine eighths of the frequency of **Do**.

$\frac{9}{8} \times 264 = 9 \times 33 = \underline{297}$ = frequency of **Re**.

So given the ratio between two consecutive notes we can find the frequency of the second note by multiplying the first frequency by this ratio.

1 Get a copy of Worksheet 6 which shows the table below. You are given the frequency (264) of the base note (Do) and ratios of frequencies of successive notes (top line). Find the frequencies of the other notes of the octave. The first one was done for you in the example above.

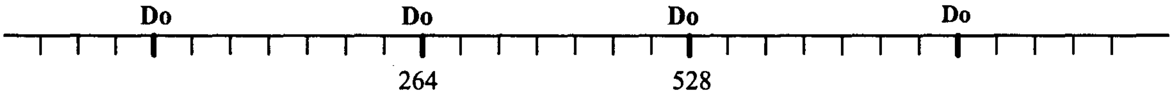
	$\frac{9}{8}$	$\frac{10}{9}$	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{10}{9}$	$\frac{9}{8}$	$\frac{16}{15}$
Do	**Re**	**Mi**	**Fa**	**So**	**La**	**Ti**	**Do**
264	297						

If you have calculated correctly you should end up with top Do having twice the frequency of bottom Do.

The ratios $\frac{9}{8}$ $\frac{10}{9}$ $\frac{16}{15}$ $\frac{9}{8}$ $\frac{10}{9}$ $\frac{9}{8}$ $\frac{16}{15}$ are the ratios between the frequencies of successive notes in the octave. **We will call them the first set of ratios.**
They can be used to get the frequency of a note from the frequency of the note before it.

Of course the notes do not stop at top Do. We can continue them.

After top Do the notes continue, Re, Mi, Fa and so on and the frequencies continue to rise.



To find the frequencies of higher notes we keep multiplying by the given ratios, starting again at the beginning of the sequence of ratios. So to get the higher **Re** we multiply the higher **Do** (528) by $\frac{9}{8}$:

$\frac{9}{8}$ $\frac{10}{9}$ $\frac{16}{15}$ $\frac{9}{8}$ $\frac{10}{9}$ $\frac{9}{8}$ $\frac{16}{15}$ $\frac{9}{8}$ $\frac{10}{9}$ $\frac{16}{15}$ . . . .

Do **Re** **Mi** **Fa** **So** **La** **Ti** **Do** **Re** **Mi** **Fa** **So** **La** **Ti** **Do**

264 . . . . 528 . . . .

2 Using the next section of your Worksheet calculate the frequencies of the notes of the higher octave until you reach Do again.

Every new Do reached should have twice the frequency of the Do before it. So you should get 1056 for your last answer.

3 What do you notice about the frequencies of your higher octave compared to the lower octave?

What you find should show you an easier way to find any notes by using the notes of the first octave. You can for example find the notes of the octave below the first octave.

4 Write down the frequencies of the notes of the octave below the first one.

THE TRUE OR NATURAL SCALE

The ratios given above are for the true or natural scale and they are so named because the ratio of each note to the base note is composed of small numbers, which we will now calculate.

The ratios between successive notes are:

$$\begin{array}{ccccccccccccccc} & \frac{9}{8} & & \frac{10}{9} & & \frac{16}{15} & & \frac{9}{8} & & \frac{10}{9} & & \frac{9}{8} & & \frac{16}{15} & \\ \textbf{Do} & & \textbf{Re} & & \textbf{Mi} & & \textbf{Fa} & & \textbf{So} & & \textbf{La} & & \textbf{Ti} & & \textbf{Do} \end{array}$$

We want to relate each note to lower **Do**.
The ratio between **Re** and **Do** is just $\frac{9}{8}$ as shown above.

The ratio between **Mi** and **Do** is $\frac{9}{8} \times \frac{10}{9}$ which cancels down by 9 and by 2 to $\frac{1}{4} \times \frac{5}{1} = \frac{5}{4}$.

For the ratio between **Fa** and **Do** we multiply the first **three** ratios together: $\frac{9}{8} \times \frac{10}{9} \times \frac{16}{15}$.
And so on for the other notes.

5 Using your Worksheet complete the table of ratios below to get the ratio of each note to bottom Do.

$$\begin{array}{cccccccc} 1 & \frac{9}{8} & \frac{5}{4} \; . \; . \; . & & & & & \\ \textbf{Do} & \textbf{Re} & \textbf{Mi} & \textbf{Fa} & \textbf{So} & \textbf{La} & \textbf{Ti} & \textbf{Do} \end{array}$$

We call these fractions the second set of ratios.
They can be used to get the frequency of a note from the frequency of the base note before it.

It is easy to check that your answers are correct. If you multiply the base frequency, 264, by each of the fractions you have found you should get the frequencies calculated earlier.

For example, $264 \times \frac{5}{4} = 330$ which is the frequency we had earlier for **Mi**.

6 Check that your answers are correct by this method.

The ratios you get are all composed of small numbers except for the first, $\frac{9}{8}$ for **Re**, and the penultimate ratio, $\frac{15}{8}$ for **Ti**.
These notes can however be simply related to the octave in another way.

1				$\frac{3}{2}$			2
Do	**Re**	**Mi**	**Fa**	**So**	**La**	**Ti**	**Do**
264	297	330	352	396	440	495	528

Note first that the note So is the mean of the frequencies of the upper and lower Do notes:

$$\frac{264+528}{2} = 396.$$

This must happen because bottom **Do** is 1, top **Do** is 2 and $\frac{3}{2}$ is the mean of 1 and 2.

Now we can extend into the next octave:

		1				$\frac{3}{2}$			2		
Do	**Re**	**Mi**	**Fa**	**So**	**La**	**Ti**	**Do**	**Re**	**Mi**	**Fa**	**So**
264	297	330	352	396	440	495	528	594	660	704	792

you can see that **Ti** is half way between the lower and upper **Mi** notes: $\frac{330+660}{2} = 495$.
This means that **Ti** is simply related to the other notes of the octave.

Similarly **Re** is half way between lower and upper **So**: $\frac{396+792}{2} = 594$:

				1				$\frac{3}{2}$			2
Do	**Re**	**Mi**	**Fa**	**So**	**La**	**Ti**	**Do**	**Re**	**Mi**	**Fa**	**So**
264	297	330	352	396	440	495	528	594	660	704	792

So all the notes of the octave can be generated from ratios of small of small numbers, the numbers 1, 2, 3, 4, 5.

EXERCISE 1

a Taking the base note as 48 (instead of 264) find the notes of the octave which starts with Do at this note. You can do this by multiplying the base note by the second set of ratios or by using the first set of ratios to get each frequency from the frequency before it.

48 ? ? ? ? ? ? ?

In the octave above this one what will be i the frequency of the top Do,
ii the frequency of Do?

In the octave below this one what will be iii the frequency of the bottom Do,
iv the frequency of Do?

b The notes indicated below start with bottom Do at 72. Find the missing numbers.

72 81 ? 96 108 ? 135 144 ? 180

c The octave shown below starts with bottom Do at 120. Two of the frequencies are wrong. Find out which two and correct the numbers.

120 135 145 160 180 200 220 240

RATIOS OF NOTES

The ratio of two notes is found by dividing the frequency of the higher note by the frequency of the lower note.

So in the octave:

Do	**Re**	**Mi**	**Fa**	**So**	**La**	**Ti**	**Do**
264	297	330	352	396	440	495	528

the ratio of **Fa** to **Re** is $\frac{352}{297}$ which cancels down to $\frac{32}{27}$.

You will find the same answer whatever octave you choose. So if you take the octave in Exercise 1, **c** you will get $\frac{160}{135}$ which also cancels down to $\frac{32}{27}$.

EXAMPLE 2

Find the ratio of **La** to **Mi**.

The ratio is $\frac{440}{330} = \frac{4}{3}$, (using the numbers above).

EXERCISE 2

Find the ratio of:

a **Ti** to **Re** **b** **So** to **Mi** **c** **La** to **Fa** **d** upper **Do** to the **So** before it

e **So** to **La** in the octave below it **f** **Mi** to **La** in the octave below it

g **So** to **Mi** in the octave below it

ANOTHER MODE

A scale does not have to start at Do.
It could begin at, say, Mi and go up seven notes from there.

Take the octave that starts with bottom Do at 24:

		1							2	
Do	**Re**	**Mi**	**Fa**	**So**	**La**	**Ti**	**Do**	**Re**	**Mi**	**Fa**
24	27	30	32	36	40	45	48	54	60	64

To find the ratio of each note to bottom Mi, which is now the base note, we divide each frequency by 30 (the frequency of lower Mi) and cancel.

7 Using your Worksheet complete the chart below in which one of the answers has been done for you:

		1	?	?	$\frac{4}{3}$	?	?	?	2	
Do	**Re**	**Mi**	**Fa**	**So**	**La**	**Ti**	**Do**	**Re**	**Mi**	**Fa**
24	27	30	32	36	40	45	48	54	60	64

8 Draw up a similar chart (using the same frequencies) to show the ratios for the mode starting at So.

THE GANDHARVA VEDA SCALE

The Gandharva Veda Scale comes from India and has 23 notes in the octave. The names of the notes and their ratio to the base note are:

1. Chhandovati	1
2. Dayavati	$\frac{25}{24}$
3. Ranjani	$\frac{16}{15}$
4. Ratika	$\frac{9}{8}$
5. Raudri	$\frac{256}{225}$
6. Krodha	$\frac{32}{27}$
7. Vajrika	$\frac{6}{5}$
8. Prasarini	$\frac{5}{4}$
9. Pritih	$\frac{81}{64}$
10. Marjani	$\frac{4}{3}$
11. Kshitih	$\frac{25}{18}$
12. Rakta	$\frac{64}{45}$
13. Sandipani	$\frac{36}{25}$
14. Alapini	$\frac{3}{2}$
15. Madani	$\frac{25}{16}$
16. Rohini	$\frac{128}{81}$
17. Ramya	$\frac{5}{3}$
18. Ugra	$\frac{27}{16}$
19. Kshobhini	$\frac{16}{9}$
20. Tivra	$\frac{9}{5}$
21. Kumudvati	$\frac{15}{8}$
22. Manda	$\frac{243}{128}$
23. Chhandovati	2

The eight notes of the octave studied earlier are contained in this scale.

9 Look carefully at the ratios and write down the notes corresponding to the eight notes of the western octave.

All the numbers in the ratios in the table above are multiples of 2, 3 and 5 and no other numbers (there are no multiples of 7 or higher prime numbers).

**The large numbers contained in these ratios are all powers of the smaller numbers.
For example 225 is $3^2 \times 5^2$ and 128 is 2^7.**

10 Express the following as a product of primes using only the prime numbers 2, 3, 5: 25, 16, 9, 8, 256, 32, 27, 81, 64, 36, 243.

11 Five of the ratios in the table above are composed of square numbers: which ones?

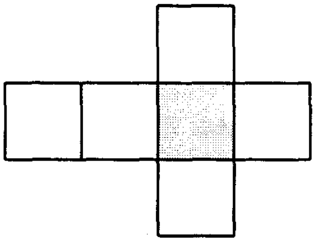
12 The ratios which are squared to get the ratios in question 11 also correspond to Gandharva Veda Scale notes.

What notes do they correspond to?

29 Nets and Networks

A net is the name given to the design on paper which, if cut out and folded up, will produce a solid, 3-dimensional shape.

So nets take us from 2-dimensional forms to 3-dimensional forms.
For example the shape below, if cut out, folded up and glued together would produce a cube.

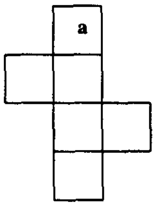


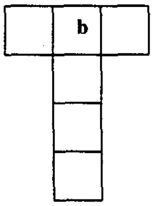
Net of a cube

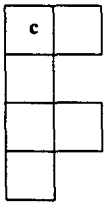
● Imagine the net being folded up, with the shaded square as base.
This is not the only design that will produce a cube, there are many others. But since a cube has 6 faces they must all be composed of 6 squares.

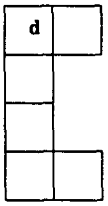
EXERCISE 1

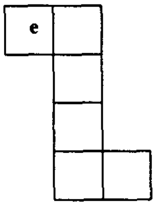
Which of these designs are nets of a cube?







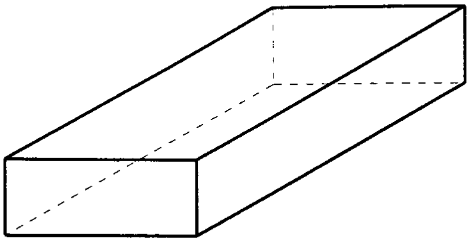


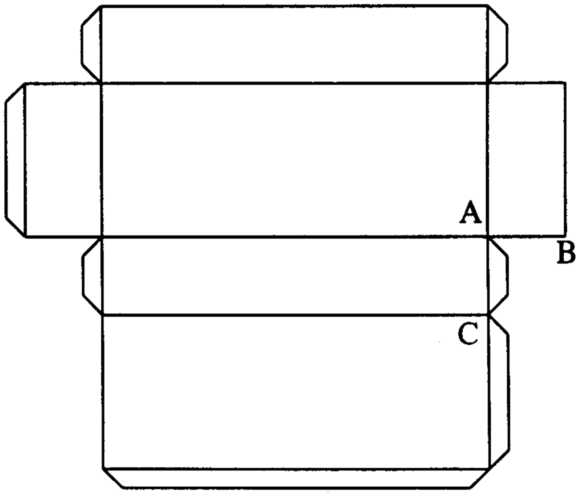


1 Draw a net of your own, different to those above, which will also make a cube. How many different nets do you think there are for a cube?
Compare your answer to those of your friends and collect together as many different nets of a cube as you can.

CUBOIDS AND PRISMS

A 'box' shape composed of rectangles is called a cuboid.





A cuboid and its net (including tabs).

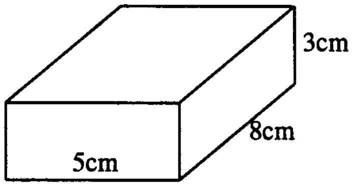
The net above has tabs included. These are glued when the net is made into a cuboid.

- Copy the cuboid shape above into your book.

If you imagine the shaded rectangle above as the base of the cuboid then you can see how the other sides will fold up to complete the shape.

Note that the lengths AB and AC, and the corresponding lengths at the other base corners, must be equal, because otherwise B will not coincide with C when it is folded up.

The cuboid shown has dimensions 5cm by 8cm by 3cm.



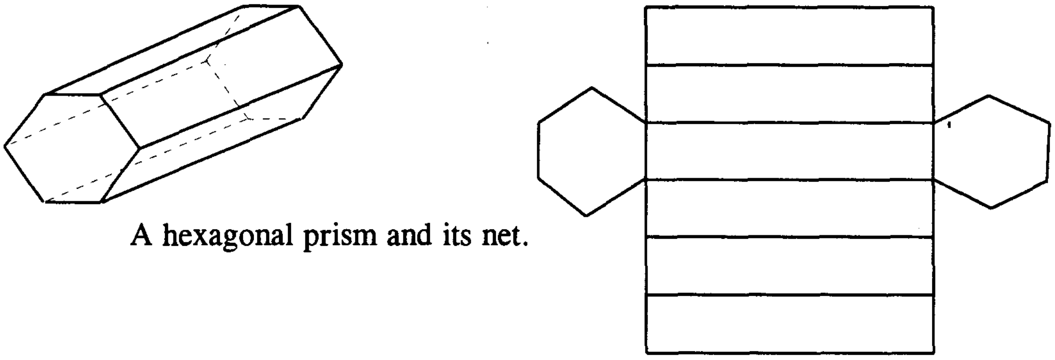
EXERCISE 2

Construct the cuboid shown above.

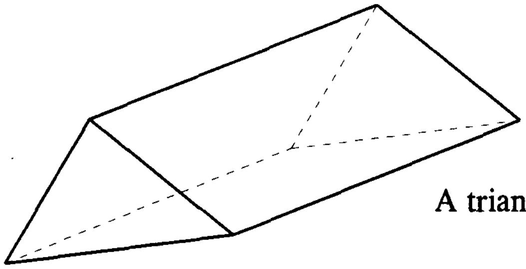
On squared paper draw accurately the net of this cuboid, including the tabs. It will be similar to the net shown above.

Cut out the net, carefully cutting round the tabs. Fold along all the lines, including the tabs. Carefully glue and attach each tab. One rectangle in the net above has no tabs- this should be the last face you attach.

A prism is a solid shape which has a constant cross-section (it is the same shape all the way through). Many pencils are examples of hexagonal prisms as they are hexagons all the way through (see below), We say it has a hexagonal cross-section.
A triangular prism is also shown below (a tent shape).



A hexagonal prism and its net.



A triangular prism.

A cuboid is also a prism because it is rectangular all the way through.

<u>EXERCISE 3</u>

Construct a triangular prism.

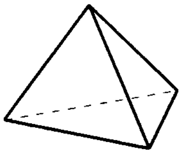
Make a sketch of the net first of all so that you have an idea of the shape.
It should have 3 rectangles and 2 triangles.

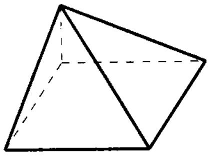
On triangular spotty paper draw the rectangular base first, 5'/<cm by 3cm, and then draw an equilateral triangle at each end of it.

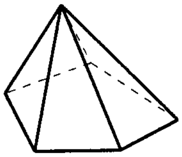
Draw the other two rectangles and put tabs where you think they are needed (there should be 5). Cut out the net, fold it up and glue it together.

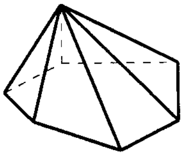
PYRAMIDS

A pyramid has a base which is a polygon and triangular faces coming from the base which meet at the vertex at the top.

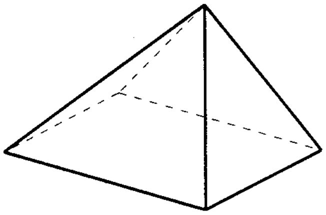

Triangular pyramid

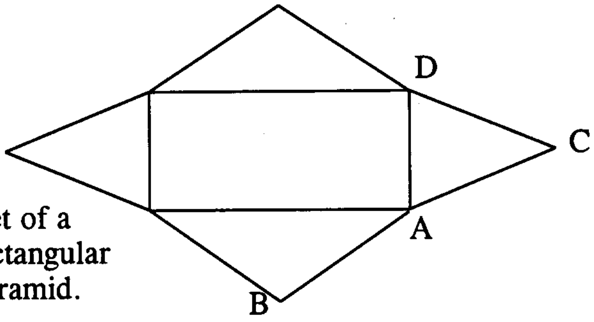

Square pyramid


Pentagonal pyramid


hexagonal pyram

Below is a rectangular pyramid and the net of a rectangular pyramid.


A rectangular pyramid.



Net of a rectangular pyramid.

In constructing these pyramids it is important that the lengths such as AB, AC and DC above are equal.

The pyramids of ancient Egypt are the only one of the "seven wonders of the world" still in existence. There are dozens of pyramids and recent research suggests that their positions on the ground somehow mirrors the arrangement of stars in the heavens.

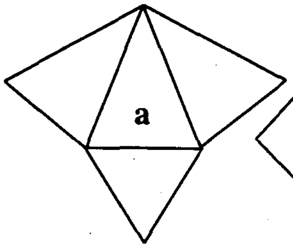
The Great Pyramid, built about 2600 BC is a square-based pyramid composed of about 2 million huge stone blocks. Many researchers have studied the remarkable accuracy of these pyramids and found surprising relationships in their dimensions.

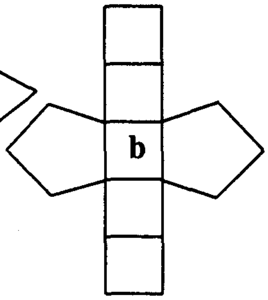
EXERCISE 4

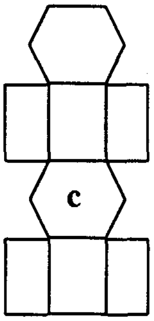
Construct a model of the Great Pyramid.
On squared paper draw a square of side 6cm.
Construct an isosceles triangle on each side of the square whose height is 4.8cm.
Draw 4 tabs where they are needed but leave one triangle free of tabs.
Cut out, fold and glue together and you have your Great Pyramid.

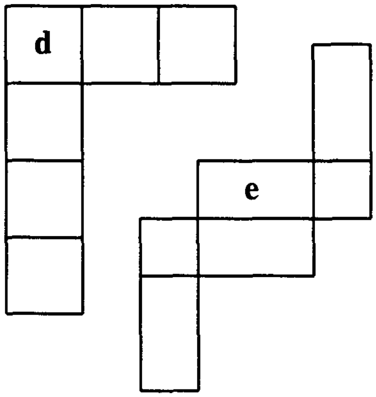
EXERCISE 5

For the 5 diagrams below decide which is the net of a solid. Give the name of the shape formed for those which are nets.

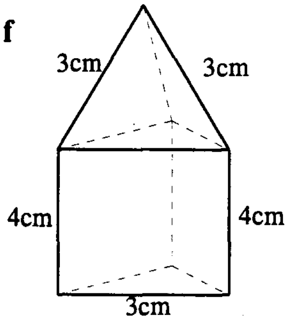








f



Draw an accurate net of the shape shown which is a triangular prism joined to a triangular pyramid composed of equilateral triangles.

Figure 1

EULER'S FORMULA

The Swiss mathematician Leonard Euler (pronounced "oiler") (1707-83) was one of the most prolific mathematicians ever: he produced hundreds of mathematical papers. He was also a rapid mental calculator- "Euler calculated without apparent effort, as men breathe, or as eagles sustain themselves in the wind".

One example of this was when two of Euler's students were calculating a series of 17 values which they then had to add. They disagreed about the figure in the 50th decimal place. Euler did the whole calculation in his head to decide which was correct.
Euler is also responsible for many of the symbols we use in mathematics today.

If you look at the shape above (Figure 1) you can count the number of edges it has (it has 12), the number of faces it has (there are 7) and the number of vertices (7).

- Check that you agree with these numbers.

We can also count the number of edges, E, the number of faces, F, and the number of vertices, V, for the other solid shapes encountered in this chapter.

EXERCISE 6

Copy and complete the table below:

Shape	Edges, E	Faces, F	Vertices, V	F + V
Figure 1	12	7	7	14
Cuboid				
Triangular prism				
Hexagonal prism				
Triangular pyramid				
Rectangular pyramid				
Pentagonal pyramid				
Hexagonal pyramid				

If you now look carefully at the columns of numbers you may see Euler's formula.

The first and last columns show the formula: that F + V is always 2 more than E.

$E+2=F+V$

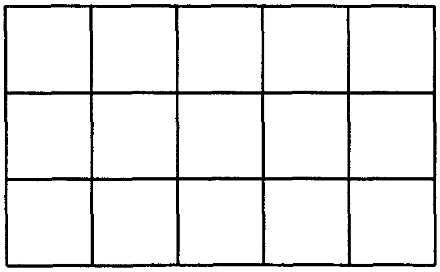
This remarkable result applies to all solid shapes composed of polygons.

Look at the last four entries in your table: they are for pyramids with bases of 3, 4, 5, 6 sides.

2 If a pyramid had a base with n sides a how many edges would it have?
b how many faces would it have?
c how many vertices would it have?

3 If you add up the answers to (b and (c above you should get the same result as adding 2 to the answer to (a , because that is what Euler's formula predicts. Check this works.

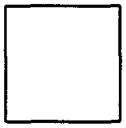
A PUZZLE



Copy the diagram above onto squared paper and divide it into 3 pieces each of which is the net for an open cubical box (a box without a lid).

NETWORKS

It is a remarkable fact that Euler's formula applies in other areas as well. A network is a system of connected lines in a plane (flat surface).



Think of the square above as a network.
The point of intersection of lines, or the end point of a line, is called a node.
The lines divide the plane into spaces called regions.

The square has 4 lines, 2 regions (1 inside and 1 outside) and 4 nodes (the 4 corners).
The lines correspond to the edges in Euler's formula, the regions correspond to faces and the nodes correspond to vertices.

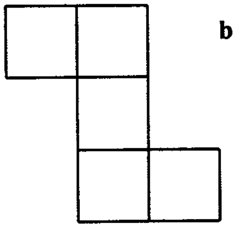
So Euler's formula would translate to Lines + 2 = Regions + Nodes.

This is confirmed for the square because 4 + 2 = 2 + 4.

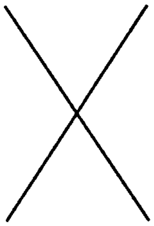
EXAMPLE 1

Show that Euler's formula applies for the following networks:

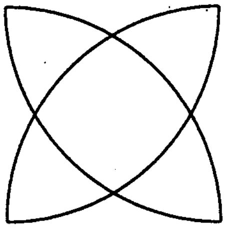
a



b



c

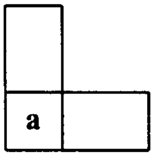


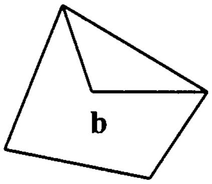
A line joins one node to the next node, so in a there are 16 lines, 6 regions and 12 nodes. In b there is a node at the end of every line so there are 4 lines, 1 region and 5 nodes. Lines need not be straight. In c there are 12 lines, 6 regions and 8 nodes.

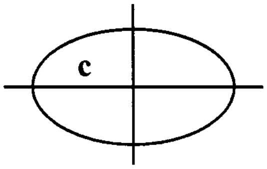
- Check that you agree with these numbers.
- Check that Euler's formula is satisfied in each of these three cases.

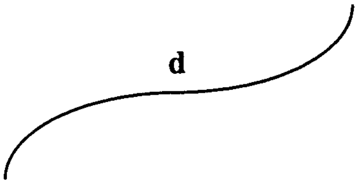
EXERCISE 7

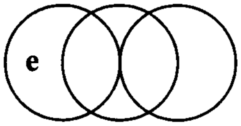
Copy and complete the table below for the 8 networks shown:

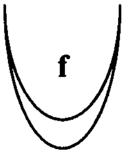

a


b

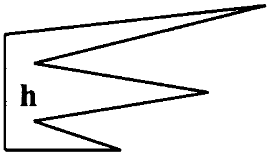

c


d


e


f


g


h

	Lines	**Regions**	**Nodes**	**Regions+Nodes**
a				
b				
c				
d				
e				
f				
g				
h				

- Confirm that Eider's formula applies in all these cases.

THE VEDIC SQUARE

You are probably familiar with the multiplication square which contains all the multiplication tables

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
2	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**
3	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**
4	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**
5	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**
6	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**
7	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**
8	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**
9	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**

This square is converted to a Vedic Square by replacing every number by its digit sum:

A	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
B	**2**	**4**	**6**	**8**	**1**	**3**	**5**	**7**	**9**
C	**3**	**6**	**9**	**3**	**6**	**9**	**3**	**6**	**9**
D	**4**	**8**	**3**	**7**	**2**	**6**	**1**	**5**	**9**
E	**5**	**1**	**6**	**2**	**7**	**3**	**8**	**4**	**9**
F	**6**	**3**	**9**	**6**	**3**	**9**	**6**	**3**	**9**
G	**7**	**5**	**3**	**1**	**8**	**6**	**4**	**2**	**9**
H	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**	**9**
I	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**

This square has many interesting and useful properties.

Worksheet A shows 9 copies of the Vedic Square.

- Colour in all the squares in the first Vedic Square which have a 1 in them.

- Form a pattern for the number 1 by joining some or all of the centres of these squares.

- Similarly obtain a pattern for the numbers 2, 3, 4, 5, 6, 7, 8, 9 on the remaining Vedic Squares.

The Vedic Square is also useful in the design of patterns.

We can do this by choosing a line of the Square and a starting point in that line.

Suppose we choose line D (4 8 3 7 2 6 1 5 9) and start at the beginning. We also choose an angle of rotation, say 90° anticlockwise.

- Take a sheet of graph paper and mark a point near the bottom left corner (you will need 2cm to the left of this).

We always start by moving to the right and the numbers in the row we have chosen tell us how many centimetres to move. (It is advisable to use a pencil for this at first)

- So now we can draw the design: first we draw a line 4cm to the right,

then turn 90° anticlockwise (to the left) and draw a line 8cm up.
then turn 90° anticlockwise and draw a line 3cm long,
then turn 90° anticlockwise and draw a line 7cm long,
and so on.

When you come to the end of the row of numbers you start again at the beginning of that row. Eventually you will return to your starting point and the design is complete.

Try another design: you will need triangular spotty paper.
Let us choose to use row D again (starting at the beginning) but now the rotation angle can be 60°.

- With the long side of your sheet at the bottom mark a dot near the middle of the bottom line.

We start moving to the right again 4cm.
Then we turn 60° to the left and draw a line 8cm long.
Then we turn 60° to the left and draw a line 3cm long.
And so on, the same as previously but with a turn of 60° instead of 90°.

- On another sheet of triangular spotty paper mark a point in the middle, and two rows down from the top of the page.

- Choose row E this time (starting at the beginning) and a rotation of 120° anticlockwise. Draw the pattern for this.

- Make a design of your own, choosing your own row and starting point and an angle of rotation. (You can also use the columns and diagonals in the Vedic Square as well as the rows)

30 Probability

In the last chapter on probability we introduced the idea of events being certain, very likely, likely, unlikely, very unlikely etc. We also learned to assign probabilities to particular events with the probability scale from 0 to 1. These can be given as fractions, decimals or percentages (in which case the scale is from 0 to 100). Certain events have a probability of 1 (or 100%) and impossible events 0. Events with an 'evens' chance of happening have a probability of 0.5 (or 50%). There are two main ways of assigning probabilities: from theory and from observation.

THEORETICAL PROBABILITIES

To calculate probabilities for equally likely outcomes, we use the following formula:

$$\boxed{\text{Probability of an event} = \frac{\text{Number of favourable outcomes}}{\text{Number of possible outcomes}}}$$

We find all the possible outcomes and from these select the favourable outcomes. It is very important that in using this formula we check that all the possible outcomes are equally likely.

When throwing a dice we tend to assume each number is equally likely to land face up. But consider a loaded dice which nearly always comes up 6! We would not be in a position to use the above formula.

EXAMPLE 1

When throwing an unbiased 20-sided die, what is the probability of throwing a prime number?

There are 20 equally likely possible outcomes. Of those, there are 8 favourable outcomes. (There are 8 prime numbers between 1 and 20; 2, 3, 5, 7, 11, 13, 17 and 19. So the probability is $\frac{8}{20}$ or $\frac{2}{5}$. (It is preferable to simplify the fraction where possible.)

EXERCISE 1

1) If one card is picked at random from a pack of 52 playing cards, what is the probability it is:
a a Queen **b** the Ace of Spades **c** a club?

2) A number is selected at random from the set of natural numbers from 1 to 50 inclusive. Find the probability it is:
a even **b** a square number **c** a prime number **d** a multiple of 3
e a factor of 50 **f** greater than 30 **g** a triangle number **h** divisible by 6
i less than 10

3) Nine counters numbered 1, 2, 3, 4, 5, 6, 7, 8, 9 are placed in a bag. One is taken out at random. What is the probability it is **a** a '5' **b** divisible by 3 **c** less than 4?

4) A bag contains 2 green balls, 4 red balls and 7 blue balls. One ball is taken out at random. What is the probability it is: **a** green **b** red **c** blue **d** yellow?

5) An unbiased coin is tossed three times. On each occasion it comes up heads. What is the probability that the next throw will be heads?

6) An unbiased die has faces numbered 0, 1, 1, 3, 5, 5. If it is thrown what is the probability of :
a a '1' **b** a number less than 4 **c** An odd number?

7) Out of 12 girls and 8 boys, what is the probability that one person selected at random will be a girl?

8) A bag contains four 1p coins, six 2p coins, three 5p coins, five 10p coins and two 20p coins. If one coin is selected at random, what is the probability it is:
a a 2p coin **b** a 5p coin **c** a 'silver' coin?

RELATIVE FREQUENCY

Often it is not possible to calculate a probability theoretically, but it is possible to conduct an experiment or carry out a survey to find what is called the relative frequency of an event. The frequency of an event is the number of times that event occurs in a number of trials. The relative frequency of an event compares the frequency with the number of trials. It is the proportion of times the event occurs in a number of trials.

$$\text{Relative frequency of an event} = \frac{\text{Number of times an event occurs}}{\text{Number of trials}}$$

We can use the relative frequency of an event as an estimate of the probability of that event occurring, provided we do the experiment a number of times. The more times we do the experiment, the better the estimate.

● Consider the following question. What is the probability of a drawing pin landing point up if we were to throw it like a die? There are two possible outcomes, point up or point down, which are unlikely to occur equally often. It is impossible to calculate theoretically, what the probabilities would be. Apart from anything else, the results will vary according to the make of drawing pin. We could however, perform an experiment to find the relative frequencies for point up and point down, for a particular pin, and use these as an estimate of probabilities.

1 Carry out an experiment to calculate the relative frequencies of a drawing pin landing point up and point down recording your results in a tally chart. Remember to perform the experiment a number of times. Compare your results with other members of the class. It might be an idea to collect results for the class as a whole as this will give you a better estimate of the probabilities.

EXAMPLE 2

Tom does a survey to find out the colour of cars. In his survey of 50 cars, 18 were red, 13 blue, 7 white, 6 black, 5 silver and 1 pink. Using this information what is the probability that the next car he sees is: **a** red **b** white.

a The relative frequency of red cars (and hence the estimate of the probability of the next car being red) is $\frac{18}{50}$ or $\frac{9}{25}$.

b Similarly the probability of the next car being white is $\frac{7}{50}$ (or 0.14).

EXERCISE 2

1) In an experiment on drawing pins; out of 60 throws 22 landed point up. What is the probability of a drawing pin landing: **a** point up **b** point down.

2) In a survey of a class of school children, 9 out of 27 children had shoe size 4.
   a What is the probability that someone in a different class, but same year had shoe size 4?
   b What about a pupil from a different year group?

3) **a** In a car survey 45 out of 60 cars only contained one person. What is the probability the next car that was seen only contained one person? Give your answer as a fraction, decimal and percentage.
   b Do you think the results of this survey would be affected by the day you carry it out? Give reasons.

MUTUALLY EXCLUSIVE EVENTS

When considering probabilities it is very important to consider whether events are 'mutually exclusive' or not. That means that they cannot to happen at the same time.

Consider a fair die that is thrown. There are 6 possible outcomes: 1, 2, 3, 4, 5 and 6. The events 'throws an even number' and 'throws a 5' cannot happen at the same time (in one throw). They are mutually exclusive events.

The events 'throws an odd number' and 'throws a 5' can happen at the same time. They are not mutually exclusive.

Some important results in probability contain this concept. One of the most important is that:

$$\boxed{\text{The total sum of probabilities for mutually exclusive events} = 1}$$

So with a coin, for example, there are two possible outcomes: heads and tails. These can not both happen at the same time so are mutually exclusive and there are no other possibilities (unless the coin lands on its side). Therefore the probability of a head **or** a tail must be 1, which we know is true ($\frac{1}{2}+\frac{1}{2} = 1$).
Similarly with a die the total probability of a 1, 2, 3, 4, 5 or 6 is 1 ($\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$).

However, suppose we consider the probability of the die in a different way. The probability of an odd number = ½ and an even number = ½. They are mutually exclusive and no other outcomes are possible. So the total of the probabilities = 1.

Now consider the events 'prime number', 'divisible by 3' and 'square number'. These events account for all possibilities (check that you agree), but they are not mutually exclusive (the outcome '3' appears in two categories i.e. 'prime number' and 'divisible by 3') and therefore the sum of their probabilities will not be 1. ($\frac{1}{2} + \frac{1}{3} + \frac{1}{3} = 1\frac{1}{6}$).

Similarly, the events 'square number' and 'prime number' they are mutually exclusive, but not every outcome is covered. Here the sum of the probabilities is $\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$. We need to add an extra category which only includes the 'missing' outcome 6. We could add 'multiple of 6'. Then $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$.

One important special case of this is as follows:

The probability of an event **not** happening = 1 - the probability of the event happening
and
the probability of an event happening = 1 - the probability of the event **not** happening.

This is because 'an event happening' event not happening' are mutually exclusive and there are no other possibilities. An event must either occur or not occur!

This illustrates the Sutra By the Completion.

EXAMPLE 3

What is the probability of not getting a 4 when throwing a fair die?

The probability of getting a 4 is $\frac{1}{6}$, so the probability of not getting a 4 is $1-\frac{1}{6} = \frac{5}{6}$.

EXAMPLE 4

In a game of hockey the probability of winning is estimated as being 0.35 and drawing 0.2. What is the probability of losing?

There are only 3 possible outcomes and they are mutually exclusive. Therefore the probability of losing is 1-0.35-0.2 = 0.45.

EXERCISE 3

1) The probability of John being late for school is 0.15. What is the probability he will not be late for school?
2) In a rugby game the probability of the home team winning is estimated as $\frac{2}{3}$ and losing $\frac{1}{4}$. What is the probability of drawing?
3) The probability of winning the national lottery is $\frac{1}{13{,}983{,}816}$. What is the probability of not winning the lottery?!
4) There are 7 red, 6 blue, 4 yellow, 3 white and 2 purple counters in a bag. If one is taken out at random what is the probability it is:
   a blue
   b not blue ? Show how to calculate the answer in 2 ways and check you get the same answer.
5) In a survey over 30 days a bus was late 4 times and cancelled twice. Using this information, estimate the probability it is on time tomorrow.
6) In a particular country they found that in the past 50 years it had rained an average of 22 days in April.
   a What is the probability it will rain there next April 7th?
   b What is the probability it will not rain there next April 7th?

EXPECTED NUMBER

We can use probability to estimate the number of times an event can occur.

EXAMPLE 5

If I throw a die 240 times how many threes would I expect?

For any one throw: $P(3) = \frac{1}{6}$ (P(3) is a useful way of writing "the probability of getting a 3".
For 240 throws I would expect $240 \times \frac{1}{6} = 40$ threes.

EXAMPLE 6

The probability a computer has a particular virus is 0.003. A large company has 10,000 computers. **a** How many computers would you expect to have the virus?
b How many computers would you expect not to have the virus?

a P(virus) = 0.003. Expected number with the virus = $0.003 \times 10{,}000 = 30$

b P(no virus) = 1-0.003 = 0.997. Expected number without virus = $0.997 \times 10{,}000 = 9{,}970$

EXERCISE 4

1) If 20% of cars are red. How many cars would you expect to be red in a random survey of 150 cars.

2) The probability of passing a test is $\frac{4}{5}$. If 80 people took the test how many would you expect to **fail**?

3) How many sixes would you expect if you threw a fair die 90 times?

4) In a marathon the probability of **not** finishing is 0.15. In a race with 2000 people, how many runners would be expected to finish?

5) In a survey 30 people out of 150 had blonde hair. In a similar survey the next day 400 people were involved. How many do you think had blonde hair?

We can also try some of these ideas experimentally. Try the following activities. In each case say beforehand what you would expect and then carry out the experiment. Record your results in a frequency table.

2 Toss a coin 100 times and record which side comes up.

3 Throw a die 60 times and record the score.

4 Make a square spinner with 4 different colours and spin it 60 times and record the colour.

We can use the results to check if our spinner, die or coin is fair. If we get a result a long way from what we would expect then we may suspect it is not fair. The more we repeat the experiment the more certain we can become.

COMBINED EVENTS

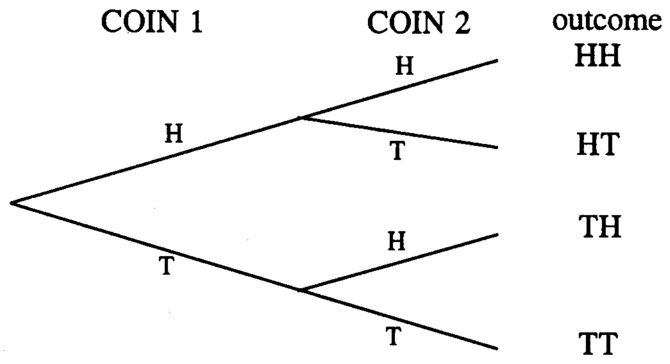
We now need to consider more complicated examples in probability, when we have more than one event, for example tossing two or three coins or throwing two dice. As before, one of the most important things to do initially is to make a list, table or diagram of all possible outcomes. We need to make an exhaustive list of the possible outcomes- that is all possible outcomes. Outcomes can be given as a list, on a diagram or in a table. As before, we have the same ways of assigning probabilities: from theoretical calculations or observed relative frequencies. Once we know the relevant probabilities we can make predictions for the expected number of outcomes of a particular event.

Making a List

Perhaps the simplest method of deciding on all possible outcomes is to make a list. This is not necessarily the most reliable method, as it is easy to miss some possibilities. If, for example, we wanted to list all possible outcomes of tossing a coin and throwing a die at the same time we might write: H1 H2 H3 H4 H5 H6 T1 T2 T3 T4 T5 T6,
where H stands for heads, T for tails and the numbers 1 to 6, the score on the die.

A Tree Diagram

A very popular and useful way of making a list of all possible outcomes is to use a tree diagram. If, for example, we wanted to show the possibilities when tossing two coins we might write the following:



We then need to consider whether the outcomes are mutually exclusive- which they are, as they can not all happen at the same time and whether they are all equally likely. If we assume the coin is fair all the outcomes are equally likely. There are four of them and then using the formula at the beginning of the chapter,

$$\text{Probability of an event} = \frac{\text{Number of favourable outcomes}}{\text{Number of possible outcomes}}$$

we are able to assign probabilities.

5 What is the probability of

a Two heads b One head and one tail c Two tails?

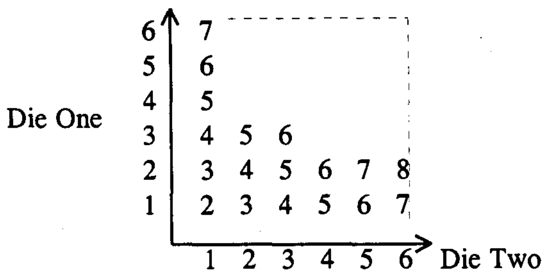
6 If you tossed two coins 60 times how many times would you expect the following results?

a Two heads b One head and one tail c Two tails

7 Toss two coins 60 times and record your results in a frequency table. Were your answers to question 6 right?

A Two-Way Table

For certain situations a two way table is an excellent way of finding out all the possible outcomes. Consider the possibilities when throwing two dice. Copy and complete the following table which shows the total score of the two dice:



8 a How many possibilities are there?
b Are they all equally likely?
c How many different total scores are possible?
d List all possible scores and assign probabilities to each one.
e Are all total scores equally likely?
f Which is the most likely score?
g Which scores are least likely? Why is this?
h Show that the total of all the probabilities is 1.

9 If you were to throw two dice 72 times write down how many of each particular total score you would expect.
10 Throw two dice 72 times and record your results in a frequency table? Were your predictions right?

Agree on some or all of the following activities with your teacher or try some of your own. In each case decide beforehand on all possible outcomes using a relevant diagram or table. Check all outcomes are mutually exclusive and equally likely. Assign probabilities in each case and predict how many of each particular outcome you would expect.

11 Toss three coins 40 times.

12 Throw two dice 50 times, recording the difference between the two scores.

13 Throw one die and toss a coin 50 times. Try letting heads = 2, tails = 5 and adding this to the number shown on the die. Or try some variation on this for yourself.

> *"Actually, the value of $\frac{\pi}{10}$ is given . . .*
>
> *Gopibhagyamadumvrata Sringishodadhisandhiga*
> *Khalajivitakhatava Galahalarasandara.*
>
> ***It is so worded that it can bear three different meanings- all of them quite appropriate. The first is a hymn to the Lord Sri Krsna; the second is similarly a hymn in praise of the Lord Sri Shankara; and the third is a valuation of $\frac{\pi}{10}$ to 32 places of decimals! (with a "Self-contained master-key" for extending the valuation to any number of decimal places!) . . .***
>
> $\frac{\pi}{10} = 0.3141592653589793$
> $2384626433832792\ldots$

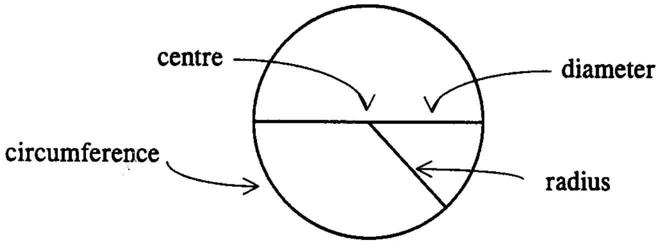
- Bharati Krsna Tirthaji
FROM "VEDIC MATHEMATICS" PAGE 363 7c

31 π

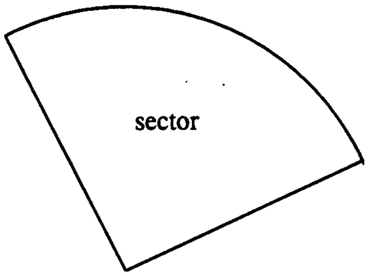
The Greek letter (pronounced `pie') is the symbol used to denote the number of times that the diameter of a circle can be wrapped around the perimeter.

The circle has often been considered the most perfect form. Its endlessness, its perfect symmetry surrounding a cental point has a special fascination. The circle and its many properties have been a source of interest and study for many years and especially the value of n, which has been calculated to billions of decimal places.

You will need to be familiar with the following words in this chapter.
The circumference of a circle is the distance around the outside of it.
The radius of a circle is a line or length of a line that goes from the centre to the circumference.
The diameter is a line from a point on the circumference through the centre to the opposite side.



A sector of a circle is a 'slice' of the circle. It has two radii and an arc.



TO ESTIMATE π

- Take a circular coin or some other round object and mark a point on the circumference. We are going to measure the length of the circumference by rolling the coin along squared paper.

So mark a point on the paper and place the point on the coin on top of it, with the coin vertical.

Carefully roll the coin along a line until the marked point on the coin comes back onto the paper.

Be very careful that the coin does not slide, but rolls.

Repeat this 3 or 4 times to get the right length and then measure the distance between the two points on the paper.

Measure the diameter of the coin and divide the distance on the paper by the diameter. You should get an answer just over 3.

The value of π is 3.1416 to 5 significant figures, and if your answer was between 3.1 and 3.2 that is good enough by this method.

In fact it does not matter what the size of the circular object was that you used, you would have obtained the same answer: because what you have found is the ratio of circumference to diameter which is the same for all circles. You may like to do this experiment again with a saucepan or similar object using string to measure the circumference.

1 How do you think you could have obtained a better value for if you did not know the answer?

π is the number of times that the diameter of a circle can be wrapped around the outside.

Its value is **3.1416** to 5 significant figures.

We will use the value $\underline{\pi = 3.14}$ in this chapter unless told otherwise.

CIRCUMFERENCE OF A CIRCLE

This leads to an important formula which expresses the meaning of π:

$$\boxed{C = \pi d \quad \text{or} \quad C = 2\pi r}$$

If C is the circumference of a circle and d is its diameter, then $C = \pi d$.

And if r is the radius of the circle then $C = 2\pi r$, because the diameter is twice the radius.

EXAMPLE 1

Find the circumference of a circle of diameter 3cm.

Since $C = \pi d$ we get $C = 3.14 \times 3 = \underline{9.42\text{ cm}}$ to 3 S.F.
We simply multiply π by the diameter.

This is an application of the *Proportionately* formula.

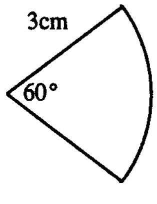
EXAMPLE 2

Find the circumference of a circle of radius 100m.

Using $C = 2\pi r$, $C = 2 \times 3.14 \times 100 = \underline{628\text{ m}}$.

EXAMPLE 3

Find the perimeter of the sector opposite.



The perimeter will be two radii, which are both 3cm, plus the length of the arc.
The 60° angle tells us that the sector is $\frac{1}{6}$ of a full circle, so if we find the circumference of the full circle and divide this by 6 we will get the length of the arc:

$$\tfrac{1}{6}C = \tfrac{2\pi r}{6} = \tfrac{2\times3.14\times3}{6} = \underline{3.14\text{ cm}}.$$

Adding the two radii to this gives: $3.14 + 3 + 3 = \underline{9.14\text{ cm}}$.

EXERCISE 1

Find the circumference of a circle with:

a radius 2cm

b diameter 5m

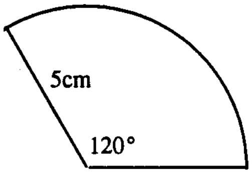
c diameter 30m

d diameter 0.2cm

e radius 0.2m

f radius 350m

g Find the perimeter of the sector:



h Find the perimeter of a quadrant (quarter of a circle) of diameter lm and angle 20°.

i Find the length of the arc in a sector of a circle of radius

It is also easy to find the diameter of a circle given the circumference.
The formula $C = 2\pi r$ can be rearranged to make r the subject. This gives $r = \frac{C}{2\pi}$.

EXAMPLE 4

Find the radius of a circle whose circumference is 86m, using $2\pi=6.3$ and giving the answer to 2 S.F.

Using $r = \frac{C}{2\pi}$, $r = \frac{86}{6.3} = \frac{860}{63}$.

Then we can use straight division:

$$\begin{array}{r|l} {}^{3} & 8\ \ {}_{2}6\ \ {}_{5}0\ .\ {}_{5}0 \\ 6 & \\ \hline & 1\ \ 3\ .\ 6 \end{array}$$

So $\underline{r = 14\text{ m}}$ to 2 S.F.

EXERCISE 2

Find the radius of a circle whose circumference is (use $2\pi=6.3$ and give your answers to 2 significant figures):

a 40m **b** 77.7m **c** 70m **d** 55m

AREA OF A CIRCLE

π appears in all sorts of unexpected places in mathematics. It would seem that since area is measured in square units and a circle is perfectly round it would be very difficult to find the area of a circle.

There is however a very simple and elegant formula using the number

.

The area, **A**, of a circle of radius **r** is:

$$\mathbf{A = \pi r^2}$$

To find the area of a circle we multiply the number n by the radius squared.

So a circle of radius 1 metre will have an area equal to π square metres (because $\pi \times 1^2 = \pi$). You will see a proof of this formula later.

EXAMPLE 5

Find the area of a circle of radius 5m.

Using $\pi = 3.14$ and since $r=5$ we get $A = 3.14 \times 5^2 = 3.14 \times 25$.
So, using the moving multiplier method, the area is $\underline{78.5\ m^2}$ to 3 S.F.

Note that although this formula is exact we are using an approximate value of so the answer is only as accurate as the value of π.

EXAMPLE 6

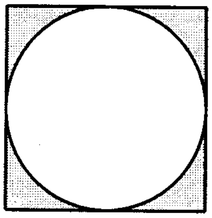
Find the area of the sector in Example 3 to 1 decimal place.

We can find the area of the full circle and divide that by 6:

$A = 3.14 \times 3^2 = 3.14 \times 9 = 28.26$.
$28.26 \div 6 = \underline{4.7\ cm^2}$ to 1 D.P.

EXAMPLE 7

Find the area shown shaded given that the side of the square is 10m.



The required area is **the area of the square minus the area of the circle.**

The area of the square is $10^2 = 100$.
The area of the circle is $\pi \times 5^2 = 3.14 \times 25 = 78.5$.

So the shaded area is 100 - 78.5 = 21.5 m².

EXERCISE 3

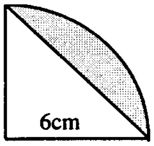
Find the area, to 3 S.F., of a circle of radius:

a 2m **b** 20cm **c** 9m **d** 103m

e 45m **f** 12m

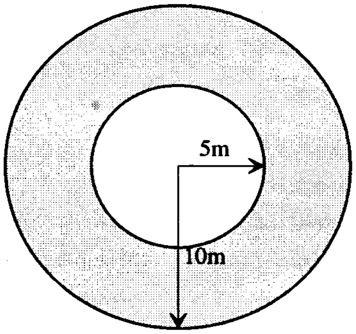
g Find the area of a quadrant of a circle of <u>diameter</u> 4m

h Find the shaded area in the quadrant below:



i Find the area of a sector of a circle of radius 0.2m and angle 30°

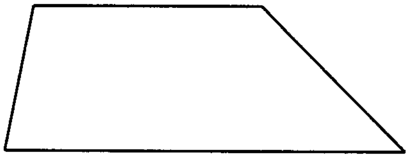
j Find the area of the shaded ring below:



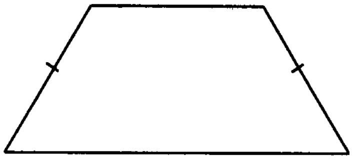
32 Volumes of Prisms and Pyramids

AREA OFF A TRAPEZIUM

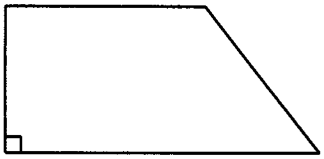
We begin by looking at the area of a trapezium. This quadrilateral has a pair of parallel sides.



There are two special types of trapezium which are frequently encountered. These are the isosceles trapezium and the right-angled trapezium:



Isosceles trapezium



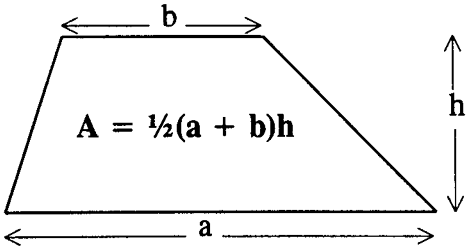
Right-angled trapezium

The isosceles trapezium has a line of symmetry so the base angles are equal and the top angles are also equal.

The area of a trapezium is given by

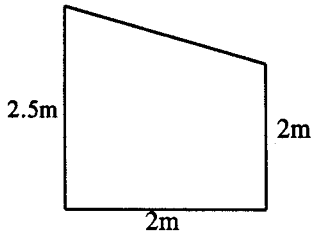
$$A = ½(a + b)h$$

where a and b are the lengths of the parallel sides and h is the height of the trapezium, or the distance between the parallel sides.



EXAMPLE 1

Find the area of the cross-section of a garden shed shown below.



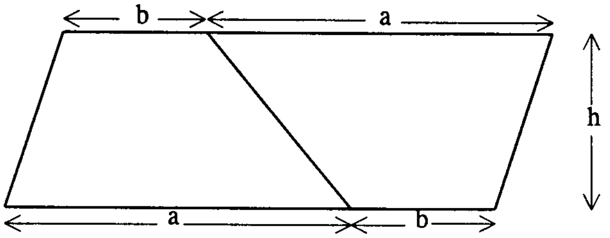
We recognise here the right-angled trapezium with its parallel sides vertical.
So in the formula a=2 and b=2.5 (or a=2.5 and b=2, it makes no difference which you choose) and h=2m.

Substituting these numbers into the formula we get

$$A = \tfrac{1}{2}(2 + 2.5)\times 2.$$

Note that the ½ cancels with the 2 here, leaving $A = 2+2.5 = \underline{4.5\ m^2}$.

The formula for the area of a trapezium is quite useful and can be proved to be true as follows. The trapezium shown above with parallel sides a and b and height h, can represent all trapeziums. In the proof we simply put two of these together but with the second one upside down:



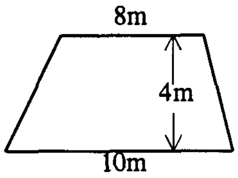
The shape formed by the two together is a parallelogram and we know that the area of a parallelogram is **base × height**.
So the area of the whole figure is (a + b)h and therefore the area of one trapezium must be ½(a + b)h.

We will be looking at proofs in more detail later on but this proof illustrates one important use of algebra in providing formulas.

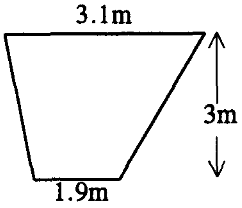
EXERCISE 1

Find the area of the following trapezia:

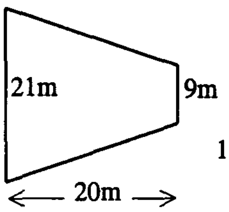
a



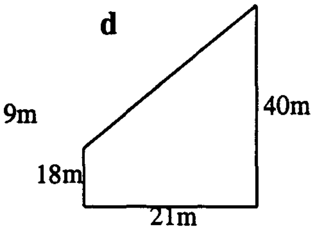
b



c

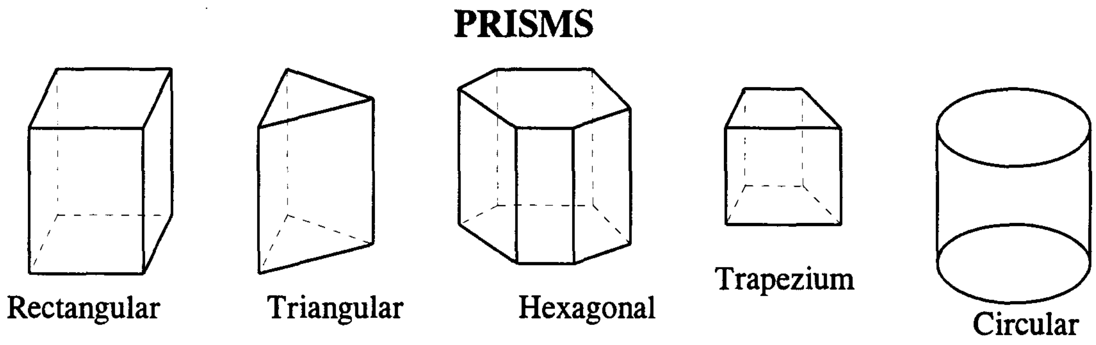


d



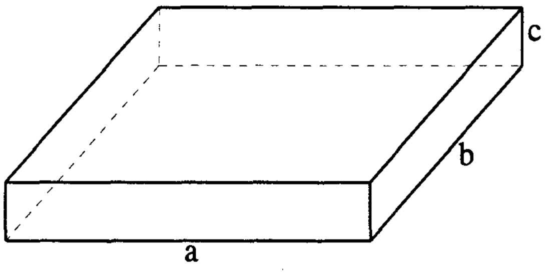
VOLUME OF A PRISM

We have seen that there are various types of prism, depending on the shape of its cross-section. The solids below are all prisms, with different shapes of cross-section.



A rectangular prism is also called a cuboid and a prism with a circular base is usually called a cylinder.

A formula for the volume of a cuboid is **V = abc** because we know that to find the volume of a cuboid we multiply the length, width and height together:



V = abc

But since the area of the base here is ab we could also write the volume as:

Volume = area of cross-section × height

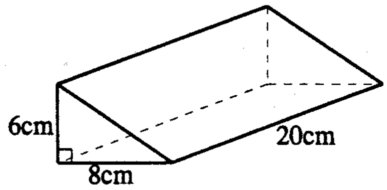
And in this form the formula gives the volume of any prism.

The volume of a prism is given by:

V = area of cross-section × height

EXAMPLE 2

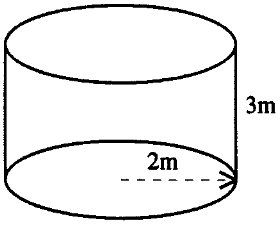
Find the volume of the triangular prism shown below.



The area of the **triangular** cross-section is ½×8×6 = 24.
So the volume is cross-sectional area × height = 24 × 20 = <u>480 cm³</u>.

EXAMPLE 3

Find the volume of a cylinder with a radius of 2m and a height of 3m.



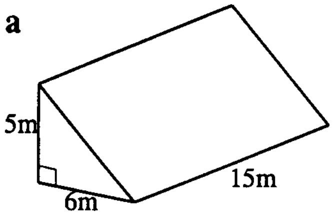
Since the area of a circle is πr^2 the volume of a cylinder is $\pi r^2 h$, where h is the height of the cylinder.

So the volume of this cylinder is $3.14 \times 2^2 \times 3$ = <u>37.68 m³</u>.

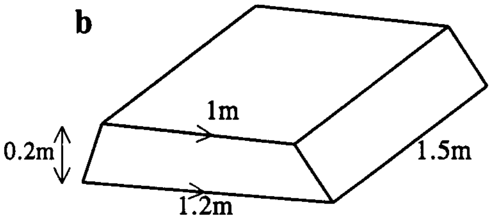
EXERCISE 2

Find the volume of the following prisms (take π=3.14):

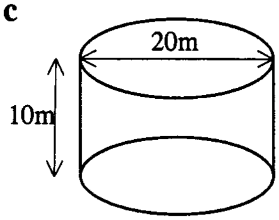
a



b



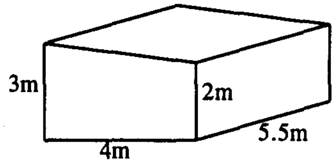
c



d A pentagonal prism has a cross-sectional area of 35m² and a length of 3.5m. What is its volume?

e A circular pipe has an internal diameter of 10cm. What volume of water can be held in a 10m length of pipe (give your answer in litres: 1000cm³ = 1 litre)?

f Find the volume of the garden shed shown opposite.



g A sheet of cardboard has sides of 62cm and 68cm. A square of side 10cm is cut from each corner of the sheet to leave a shape which is the net of an open box. Find the volume of the box.

h Six cylindrical tins are packed into a rectangular carton 30cm by 20cm by 20cm. If the tins have a radius of 5cm and a height of 20cm find the volume of the unused space in the carton.

VOLUME OF A PYRAMID

Worksheet 8 gives the net of a square-based pyramid.

- Cut out and construct the pyramid. With two friends put one pyramid with its square base on the table and add the other two pyramids to it to form a cube.

1 What is the volume of the cube?

2 What is the volume of one pyramid?

Since 3 pyramids make up the cube, the volume of the pyramid is one third of the volume of the cube.

And since the volume of the cube is base areaXheight, the volume of the pyramid is

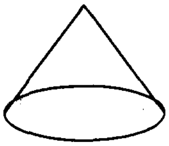
$$\tfrac{1}{3}\text{base area} \times \text{height.}$$

This is in fact a general formula for the volume of any pyramid, whatever shape of base it has.

The volume of a pyramid is given by:

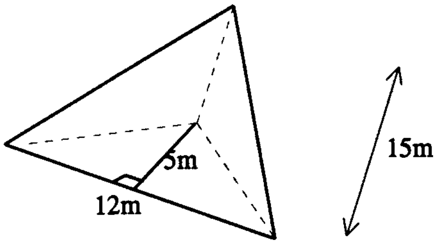
$$V = \frac{1}{3} \text{ base area} \times \text{height}$$

Since a cylinder is also a prism we find that this formula for the volume of a pyramid also applies to a cone:



EXAMPLE 4

Find the volume of the triangular pyramid below which has a height of 15m.

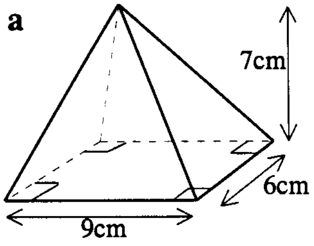


The area of the triangular base is ½base×height = ½×12×5 = 30.
The volume is $\frac{1}{3}$base area×height = $\frac{1}{3} \times 30 \times 15 = \underline{150\ m^3}$.

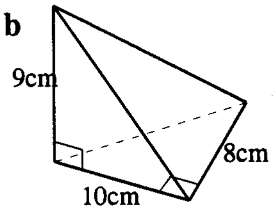
EXERCISE 3

Find the volume of the following pyramids (use π = 3.14):

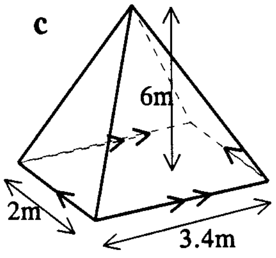
a



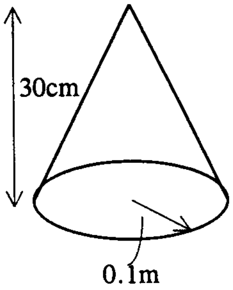
b



c



d



e A pyramid has a hexagonal base of area 96m2 and a height of 76m. What is its volume?

f The Great Pyramid of Egypt has a square base of side 440 megalithic yards and a height of 280 megalithic yards. Find its volume to the nearest cubic megalithic yard.

33 Parabolic Curves

REVISION

We have seen how to draw straight line graphs using either of two methods.

EXAMPLE 1

To draw the graph of y = 3 - 2x we could use the method of completing the triangle:

$$y = \overset{\text{intercept on y-axis}}{3} \; \overset{\text{gradient}}{-2}x$$

The 3 gives the point where the line crosses the y-axis and the coefficient of x gives the slope or gradient.
So we draw the line by marking the point at 3 on the y-axis, then move one unit to the right and 2 units down (down because the gradient is a negative number) and put another mark.
Then draw the line through these two points, extended across the page.

1 Take a sheet of graph paper and fold it in half. On one half, with the origin somewhere near the middle draw the x-axis from -3 to 3 with 2cm equal to 1 unit and the y-axis from -4 to 5 also with 2cm equal to 1 unit. Draw the graph of y = 3 - 2x.

The second method of drawing graphs was to substitute a value of x into the equation and then work out the y value. Putting x-3 for EXAMPLE into y = 3 - 2x we get y=-3.
We then plot the point (3,-3).

● Check that the line you have drawn goes through the point (3,-3).

With this method we need at least two points in order to draw the line, so we might also choose x=-1 and substituting this into the equation gives y=5. So your line should also go through (-1,5). It is usual however to find a third point to check that the first two are correct.

EXERCISE 1

a On the same axes draw the graph of **$y = \frac{1}{2}x - 1$**. Use the completing-the-triangle method an check it is right by substituting two values of x to get the coordinates of two points which should lie on the line.

❋ Check that the two lines you have drawn intersect at the point (1.6,-0.2).

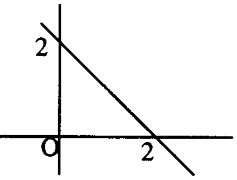
b Draw the line **$y = 3x - 1$**. This time use the substitution method to plot two points, then dra a line through them, extending the line right across the page.
Check that your line is correct by the completing-the-triangle method.

c Write down the coordinates of the points of intersection of **$y = 3x - 1$** with the other two graphs on the page. Write your answer on the graph paper next to the point.

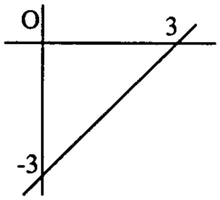
d To draw the graph of **$2y = 3x + 4$** we can first divide both sides of the equation by 2 to get $y = 1\frac{1}{2}x + 2$. Draw this line on the same page using substitution and then check with the ot method.

Write down the equation of each of the lines shown below:

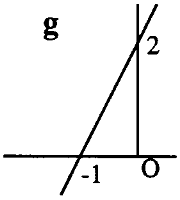
e



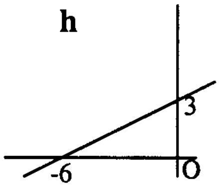
f



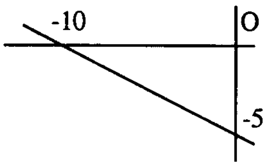
g



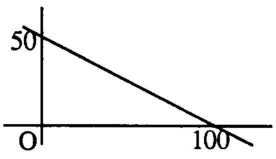
h



i



j



A THIRD METHOD- BYALTERNATE ELIMINATIONAND RETENTION

Another useful method involves using the Sutra *By Alternate Elimination and Retention.*
We first put x=0 in the equation of the line to get the intercept on the y-axis.
Then we put y=0 to get the intercept on the x-axis.

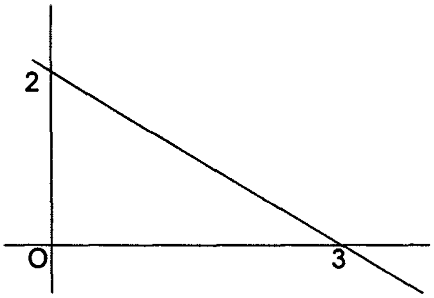
So for **$y = 3 - 2x$** (which is the first line we drew in this chapter): if x=0, y=3 and if y=0, $x=1\frac{1}{2}$.
This means the line crosses the x-axis where $x=1\frac{1}{2}$ and crosses the y-axis where y=3.

- Check that this is correct on your graph.

EXAMPLE 2

Sketch the graph of **$2x + 3y = 6$**.

If $x=0$ then $y=2$.
If $y=0$ then $x=3$.
So the intercepts are at $x=3$ and $y=2$ and the graph is:



Note that the question in this EXAMPLE asks for a sketch.

This means that it is not necessary to use graph paper or to number along each axis.
A sketch can be done on any paper: the axes are drawn with a ruler but not numbered.
The points that the line goes through can then be approximately plotted and the line is drawnalso with a ruler.

The next set of graphs will be drawn with different scales on the x and y axes. Sometimes this is appropriate in order to show the graph well.

EXERCISE 2

Draw a new set of axes on your graph paper. The origin is to be in the same place and the x-axis is the same as before, but the y-axis should be numbered from -8 to 10 with 1cm equal to 1 unit. Use the third method described above to draw the following graphs:

a $y = 3x - 6$ **b** $2x - 3y = 6$ **c** $2x = y - 4$ **d** $2y = 2x - 3$

Sketch the following graphs (do not use graph paper):

e $3x + 4y = 6$ **f** $2x - 5y = 20$ **g** $x + 2y = 50$ **h** $2x - y + 100 = 0$

PARABOLAS

So far we have only drawn graphs which are straight lines.
Equations like $y = x^2$
$y = 2x^2 - 3x + 4$ etc. do not give straight lines, but curves.
They form a curve called a **parabola**.

The parabola is an important mathematical curve. An object thrown into the air will (unless it is thrown vertically upwards) follow a curved path which is parabolic.
Mathematically the parabola has many interesting and important properties.

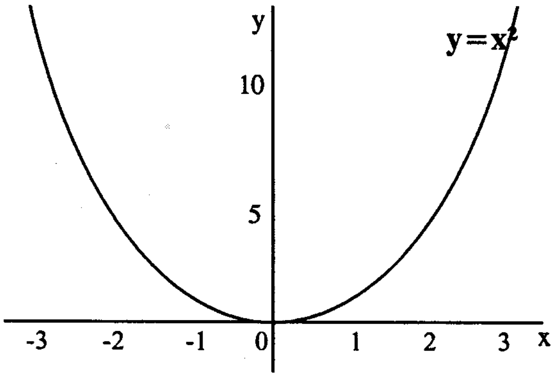
A straight line can be drawn by knowing the position of two points on it, but for curves a number of points are needed: the more points, the more accurate is the drawing.

EXAMPLE 3

Draw the graph of $y = x^2$.

We substitute a series of values of x and get the corresponding value of y.
Each x,y pair gives a point on the graph.

Putting x=3 we get $y=3^2=9$, so we plot the point (3,9).
Then we put x=2, 1, 0, -1, -2, -3 and find y for each one.
We get the points (3,9), (2,4), (1,1), (0,0), (-1,1), (-2,4), (-3,9) which we plot.
Finally we draw a smooth curve through the points:



• On a new section of your graph paper draw axes with the same scales as in EXERCISE 2. Plot the points above.

2 Find the y value corresponding to x=2½ and plot another point on the graph.

3 Similarly find y values for x=1½, ½, -½, -1½, -2½ and plot five more points.

These should fit in with the points already plotted.

• Carefully draw a smooth curve through the points like the one shown above.

Write $y = x^2$ next to the line.

EXERCISE 3

On the same axes as above draw the graph of:

a $y = x^2 + 4$ Start with x=2, which gives y=8, so you can plot (2,8).
Then x=1½, 1, ½, 0, -½, -1, -1½, -2, -2½. Each time you find y plot the point immediately. Try to do all the calculations mentally.

b $y = x^2 - 5$ Use values of x from 2½ to -2½ in steps of ½. Label both graphs.

You should find that the first curve in the above exercise is the same as $y = x^2$, except that it is **shifted 4 units up**.

Similarly the second curve is $y = x^2$ **shifted down 5 units**.

This shows that:

the graph of $y = x^2 + a$

is the same as the graph of $y = x^2$ but shifted **a** units up if **a** is positive and **a** units down if **a** is negative.

EXERCISE 4

Number a new set of axes, the same as the last ones.
Draw the following graphs:

a $y = x^2$ **b** $y = -x^2$ **c** $y = 8 - x^2$ **d** $y = -3 - x^2$

The following points are worth noting about these graphs:

1. $y = -x^2$ is the same as $y = x^2$ but upside down.
2. All equations with **minus** x^2 appear to be upside down.
3. Note that in **c** and **d** above the numbers **8** and **-3** shift the graph of $y = -x^2$ 8 units up and 3 units down in a similar way to that observed in the previous exercise.

EXERCISE 5

Number a new set of axes with the x-axis going from -6 to 6 and 1 cm for 1 unit, and the y-axis from -40 to 50 and 2cm for 10 units.

Draw the following graphs (do the calculations mentally as far as possible):

a $y = x^2$ **b** $y = 2x^2$ **c** $y = 3x^2$ **d** $y = \frac{1}{2}x^2$

e $y = -2x^2$ **f** $y = 10 - 2x^2$

The following points can be noted about these graphs:

1. Changing the coefficient of x^2 changes the shape of the graph: $y=2x^2$ and $y=3x^2$ are narrower than $y=x^2$ and $y=\frac{1}{2}x^2$ is less narrow.
2. **e** and **f** show, as before, that when the coefficient of x^2 is negative the curve is upside down.
3. As expected **f** is the same as **e** but shifted up 10 units.

EXERCISE 6

Number a new set of axes the same as those in Exercise 2, 3, 4.
Draw the following graphs:

a $y = x^2 - 2x$ **b** $y = x^2 + x + 1$ **c** $y = 1 + 2x - x^2$

d $y = -x^2 - 2x + 7$ **e** $y = x^2 - 6x + 3$

EXERCISE 7

Draw new axes both with 2cm equal to 1 unit and with x going from 0 to 4 and y from -6 to 2.
Draw the following graphs:

a $y = 2x^2 - 8x + 2$ **b** $y = x - \frac{1}{2}x^2$ **c** $y = 2x^2 - 3x$

d Write down (correct to 1 decimal place) the coordinates of the two points where graph **a** meets graph **b** .

e Write down (also to 1 d.p.) the coordinates of the points where graph **b** meets graph **c**.

Every parabola has a **turning point**, where it has its lowest or highest point.

f Write down the coordinates of the turning point for graphs **a** , **b** and **c**.

34 Sequences

We will begin by reviewing the sequences we learned about in Book 2. Then we will look at some slightly more difficult sequences. If you remember, we examined sequences that increased by an equal amount each time. This regular increase told us what number goes before the 'n' in the formula. Then we had to make by adding or subtracting a number, depending on where the sequence started from.

EXAMPLE 1

Find a formula for the 'nth' term in the following sequences:

a 5, 10, 15, 20, 25.....

b 1, 4, 7, 10, 13.....

c $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}$....

a This sequence goes up by 5 each time. It also starts at 5, so the formula will be 5n. A quick check reveals we can generate this sequence using the formula 5n.

b This sequence goes up by 3 each time so contains 3n. However it does not start at 3 but 1, so the formula will be 3n-2.

c With a sequence involving fractions we consider top and bottom separately. The formula will be $\frac{n}{2n+1}$.

Find a formula for the nth term of the following sequences:

EXERCISE 1

a 3, 6, 9, 12, 15.....

b 11, 22, 33, 44, 55.....

c 1, 2, 3, 4, 5.....

d 2, 7, 12, 17, 22.....

e 4, 5, 6, 7, 8.....

f 6, 10, 14, 18, 22.....

g 0, 2, 4, 6, 8.....

h 2, 5, 8, 11, 14.....

i $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}$....

j $\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \frac{5}{15}$....

k $\frac{1}{3}, \frac{2}{8}, \frac{3}{13}, \frac{4}{18}, \frac{5}{23}$....

l $\frac{2}{4}, \frac{4}{6}, \frac{6}{8}, \frac{8}{10}, \frac{10}{12}$....

We will now look at some different kinds of sequences.

SQUARE NUMBERS

Let us now consider the sequence of numbers known as the 'square numbers'. If you remember the sequence is as follows:

1, 4 , 9, 16, 25, 36......

1 Copy this sequence and add the next 6 terms.

Now let us consider the difference between each term as we did in Book 2.

$$\begin{array}{ccccccccccc} \mathbf{1} & & \mathbf{4} & & \mathbf{9} & & \mathbf{16} & & \mathbf{25} & & \mathbf{36}\,.\,.\,.\,. \\ & +3 & & +5 & & +7 & & +9 & & +11 & \end{array}$$

2 Continue this pattern up to the 12th square number: 3, 5, 7, 9.........

3 As you can see the increase is not the same every time, but there is a pattern. Write down a description of this pattern.

We can now consider the differences of the differences!

$$\begin{array}{ccccccccccc} \mathbf{1} & & \mathbf{4} & & \mathbf{9} & & \mathbf{16} & & \mathbf{25} & & \mathbf{36}\,.\,.\,.\,. \\ & +3 & & +5 & & +7 & & +9 & & +11 & \\ & & +2 & & +2 & & +2 & & +2 & & \end{array}$$

As you can see the differences of the differences is always 2.

- Check that you agree with this up to the 12th square number.

In fact there is general rule, that if the differences of the differences is constant, then the formula for the nth term will contain n^2.

We can also tell from the differences of the differences what number goes in front of n^2. We can do this by halving the number we get. In this case we got 2, half of 2 is 1 so the formula will be $1n^2$ or just n^2. If the differences of the differences is 6, then the formula will contain $3n^2$.

Formulas containing n^2 can get quite complicated. They may contain terms with n^2, n **and** numbers. For example, you could have a formula $3n^2+4n-5$. In this chapter, we will only consider formulas which are quite near to n^2 or some multiple of n^2 as in the examples below.

EXAMPLE 2

Find formulas for the nth term of the following sequences:

a 3, 12, 27, 48, 75.....
b 3, 6, 11, 18, 27.....
c 5, 11, 21, 35, 53.....

a In this sequence, the differences of the differences is 6. So we know the formula contains $3n^2$. Then we see how close we are by trying to generate a sequence from $3n^2$ and we get 3, 12, 27, 48, 75....., so the formula is just $3n^2$.

b The differences of the differences is 2, so the formula contains $1n^2$ or just n^2. However when we try and generate the sequence from n^2 (1, 4, 9, 16, 25.....) we find we are 2 short each time, so the formula is n^2+2.

c The differences of the differences is 4, so the formula contains $2n^2$. Generating a sequence from $2n^2$ gives 2, 8, 18, 32, 50....., which is 3 less than what we want each time, so the formula is $2n^2+3$.

EXERCISE 2

Find formulae for the nth term of the following sequences:

a 1, 4, 9, 16, 25.....
b 2, 5, 10, 17, 26.....
c 0, 3, 8, 15, 24.....
d 2, 8, 18, 32, 50.....
e 5, 20, 45, 80, 125.....
f 5, 8, 13, 20, 29.....
g 4, 16, 36, 64, 100.....
h 5, 14, 29, 50, 77.....
i 3, 9, 19, 33, 51.....
j 1, 13, 33, 61, 97.....
k 10, 25, 50, 85, 130.....
l -1, 8, 23, 44, 71.....

CUBE NUMBERS

Consider the pattern for cube numbers we found in Book 1:

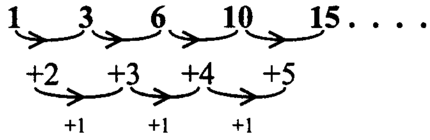
1, 8, 27, 64, 125.....

4 Copy the above list into your book. Add the next two numbers in the sequence. Write down the differences between the numbers you have written down. Then find the differences between these numbers. Find the differences between these numbers! Are they the same?

You should have found that the differences of the differences of the differences was the same each time, in this case 6. You may remember the formula for this sequence is n^3. For any sequence containing n^3 (or a multiple of n^3), only the differences of the differences of the differences is the same!

TRIANGULAR NUMBERS

Do you remember the triangular number series from Book 1?
5 Copy the following series of triangular numbers including the 'differences' and extend the sequence until you have all the triangular numbers less than 100:



✸ What do you notice? What clue does this give us about the formula? The differences of the differences is 1, indicating the formula for the nth term contains $\frac{1}{2}n^2$.

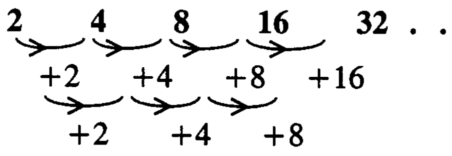
6 Write out the series of numbers that the formula $\frac{1}{2}n^2$ generates. Can you see how to get from this series to the series of triangular numbers?

You may remember the formula for triangular numbers was **$\frac{1}{2}n(n+1)$**. What happens if we multiply out the brackets? We need to multiply $\frac{1}{2}n$ by both n and 1. This gives us **$\frac{1}{2}n^2+\frac{1}{2}n$**. So this is another version of the same formula, though it is more usual to see **$\frac{1}{2}n(n+1)$**. Check that both formulas generate the sequence we want: 1, 3, 6, 10, 15, 21.....

This is quite a hard formula to derive, but because it occurs so often in many practical situations and puzzles, it is worth remembering.

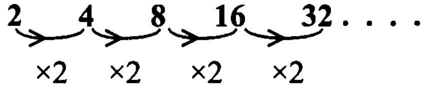
A POWER SEQUENCE

Let us have a look at another very common series:



7 a Copy out the above series and add the next five terms.
b How do you get the next number in the series?
c Explain what happens when you look at the differences between the numbers in the series, the differences between these numbers, the differences between these numbers and so on.

You should have noticed that you get the same sequence each time, so it does not look as though looking at the 'differences' is going to be a productive way of trying to find the formula for the nth term. However if we look at *All the Multipliers* we do find something constant:



We are multiplying each term by 2.

8 Copy and complete the following: for the first term we have 2,
the second term we have $2 \times 2 = 2^2$ (= 4),
the third term we have $2 \times 2 \times 2 = 2^3$ (= 8),
the fourth term we have...
the fifth term we have...
the sixth term we have...
the nth term we have...

This give us the formula for this series. We multiply by two each time, so for the nth term we need to multiply by two n times. The formula for the nth term is therefore 2".

There are some related sequences. For EXAMPLE consider the following series:

3, 9, 27, 81, 243.....

9 Copy the above series and add the next two terms. What are we multiplying by each time? What do you think the formula will be?

The formula here is 3^n.

3^n gives the 9-point circle pattern on the first page of Chapter 3 of this book: by finding the digit sums of the terms of the sequence for 3^n and joining them in order. Similarly for the other chapters.

EXERCISE 3

What will be the formula for the following sequences:

a 5, 25, 125, 625..... **b** 4, 16, 64, 256.... **c** 8, 64, 512, 4096....

d 1, 3, 7, 15, 31.... **e** 4, 6, 10, 18, 34.... **f** 0, 6, 24, 78, 240....

g 5, 17, 65, 257, 1025....

GAMES AND PUZZLES

Here are some games and puzzles for you to try. Your teacher make give you some others instead of or as well as these. In each case try and use some of the following strategies to help you solve the problems:

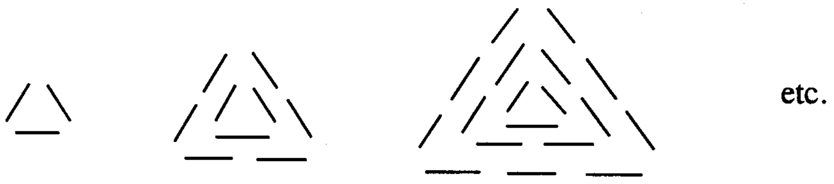
***Be systematic**
***Make a table**
***Find a general rule**
***Check regularly**
***Try some simple examples first**
***Spot patterns**
***Explain why it works**
***Find a helpful diagram**
***Use the patterns**
***Find a formula**

In looking for a general formula from particular results you are using the Vedic formula *Specific and General.*

If you do not understand the question, get stuck or can not find a formula you can ask your teacher for some hints, but try and solve as many of the problems on your own as possible. Remember to look back at some of the work already covered in this chapter to help you. Try and produce a formula each time, but remember it is the way you go about tackling the problem that is as important as coming up with a formula. Good luck!

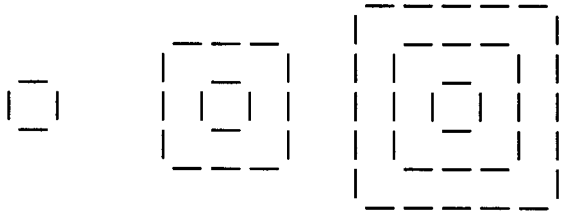
A - EXPANDING TRIANGLES

Make the following shapes with match sticks. Investigate the total number of matches used to make the shape each time.



B - EXPANDING SQUARES

Make the following shapes with match sticks. Investigate the total number of matches used to make the shape each time.



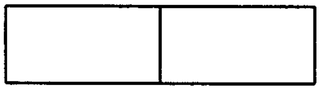
etc.

C - RECTANGLES

Imagine you have a box full of wooden building blocks the same size.

We shall start with one block.
There is one rectangle here.

Now imagine two building blocks lying next to each other.



You should find three rectangles.
What about three blocks lying next to each other?

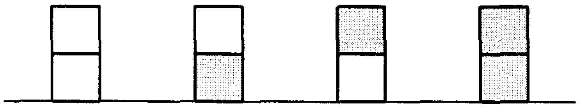
How many rectangles are there? Investigate this pattern.
Examine what happens to the number of rectangles as you increase the number of blocks.

D - THE CHESSBOARD PROBLEM How many squares are there on an 8×8 chessboard? There are not 64! (Hint: start by looking at smaller boards, 1×1, 2×2, 3×3.....etc.) Can you generalise your results for an n×n square?

E - THE HANDSHAKE PROBLEM Imagine there are a number of people in room. They all have to shake hands with everyone else. How many handshakes will there be? (Find out what happens to the number of handshakes as the number of people in the room increases. How many handshakes would there be for n people?)

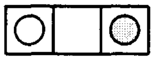
F - TOWERS You have a box of building blocks. They are all cubes. Half are black and half are white. If you had to make a tower with one block there are two different possibilities:

If you could make a tower with two blocks how many possibilities would there be?

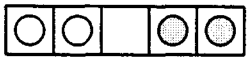


Investigate.

G - FROGS You will need counters of two different colours. Start with the following situation:



Try and swap over the two counters using 'hops' and 'slides'. Record how many it takes.

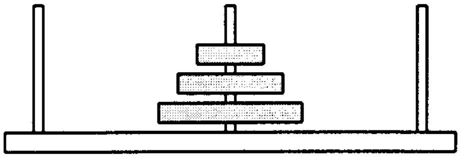


What happens if you start with two of each colour?

Try starting with more counters; 3, 4..... Record the **minimum** number of moves each time.

If you can not find a formula for the pattern, try and record your results for 'number of hops' and 'number of slides' separately.)

H - TOWERS OF HANOI This puzzle was invented by a French mathematician, Eduard Lucas and sold as a toy in 1883. There are three 'spikes', one of which has several discs on it. Each disc is smaller than the one below it. You have to transfer all the discs from one spike on to another.



The rules are:

1) Move only one disc at a time.

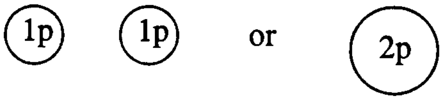
2) A disc may only be placed on a larger disc or empty spike.

3) The transfer must be completed in the smallest number of moves. Investigate the number of moves it takes as you increase the number of discs.

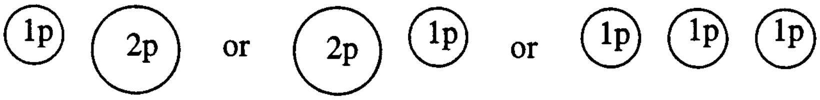
I - COINS You have a bag of 1p and 2p coins. You have to lay coins out in a row to make different amounts of money. Investigate how many ways there are of making 1p, 2p, 3p, 4p etc. Suppose that the **order** of the coins **is** important.

For example there is only one way of making up 1p:

two ways of making 2p:



but three ways of making 3p:



THE FIBONACCI SEQUENCE

The sequence you found in I above, is very like the 'Fibonacci sequence', named after the Italian mathematician Leonardo di Pisa (also named Fibonacci) who lived from about AD 1170 to 1250. He wrote a famous book, Liber Abaci, which described the Hindu-Arabic number system which we use today (at that time Roman numerals were used). Also in this book was a problem which leads to the Fibonacci sequence:

1, 1, 2, 3, 5, 8, 13, 21 . . . .

10 Copy the above sequence and find the next three numbers in the sequence. Explain how you got your answers.

You should have noticed that each number is the sum of the previous two numbers. The numbers in the Fibonacci sequence frequently occur in nature. The sunflower for EXAMPLE has 21 spirals one way and 34 the other way.

11 Copy and complete the following table for all the numbers you have found in this sequence. If you wish you may like to use a spreadsheet to do this, in which case you can easily extend the table much further (if you are not sure how to do this ask your teacher for help). The 'ratio' is the

ratio of that value divided by the previous value and the 'decimal' is the decimal equivalent of that ratio. Give the decimals to 3 D.P.

Term	Value	Ratio	Decimal
1	1	-	-
2	1	$\frac{1}{1}$	1
3	2	$\frac{2}{1}$	2
4	3	$\frac{3}{2}$	1.5
5	5	$\frac{5}{3}$	$1.\dot{6}$
6	8	$\frac{8}{5}$	1.6

12 Draw a graph with the values from the 'term' column on the horizontal axis and the 'decimal' column on the vertical axis. Describe what happens.

The ratio seems to 'settle' to a particular value. The value that this sequence approaches is THE GOLDEN RATIO. You may recall this from Book 2. In fact the value of the golden ratio is 1.6180 to 4 d.p.

A SUMMARY

Here is a summary of the sequences met so far:

SEQUENCE	FORMULA	NAME OF SEQUENCE
2, 4, 6, 8, 10 . . .	2n	Even numbers
1, 4, 9, 16, 25 . . .	n^2	Square numbers
1, 8, 27, 64, 125 . . .	n^3	Cube numbers
1, 3, 6, 10, 15. 21 . . .	$\frac{1}{2}n(n+1)$	Triangular numbers
2, 4, 8, 16, 32, 64 . . .	2^n	Powers of 2
1, 1, 2, 3, 5, 8, 13 . . .	-	Fibonacci sequence

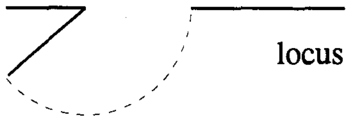
35 Loci

A locus (loci is the plural) is the path of a moving point.
As already mentioned, a projectile fired or thrown into the air travels in a parabolic curve. We say- the locus of the projectile is a parabola.

Similarly the locus of the Moon travelling around the Earth is approximately a circle. The locus of the Earth travelling around the Sun is approximately a circle.

The locus of the Moon as it moves around the Earth which is also moving around the Sun is, however, rather difficult to visualise.

The locus of a door knob as the door is opened is an arc of a circle.



locus of a door knob viewed from above

There are three things involved in any locus:

the object or point moving (the door knob in the EXAMPLE above),
the law according to which it moves (it is fixed to a rotating door)
and the locus itself which is the result of the first two.

<u>EXERCISE 1</u>

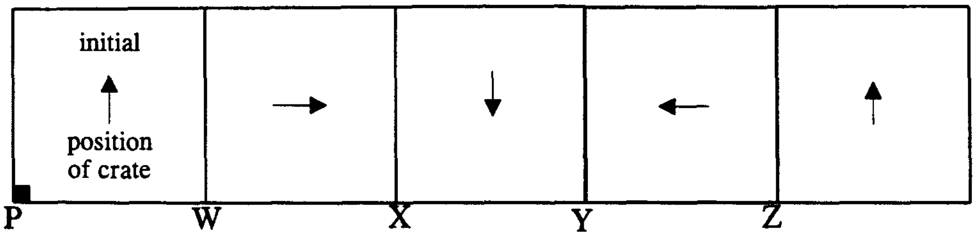
a A point is fixed on the floor and a point, P, moves on the floor so that it is always 1 metre from the fixed point. What is the locus of P?

b What will be the locus of the same point, P, if it not restricted to being on the floor?

c In your book mark two points, A and B, 4cm apart, with A above B. Mark also a point which is equidistant (the same distance) from A and B. Find and mark another point which is equidistant from A and B. How many points are there which are equidistant from A and B? Draw, with a colour, the locus of points which are equidistant from A and B.

EXAMPLE 1

A crate with a square end is being moved across a floor by tipping it over and then tipping it again. There is an arrow on the end of the crate which is initially pointing upwards, and one corner is marked P as shown:



As the crate is tipped from its first position on the left it pivots on W. The problem is to draw the locus of P.

- Copy the diagram neatly into your book. The size of the squares is not important but they should not be too small.

- Imagine the square rotating about W and mark the position of Pin the second square when the crate has been moved a quarter turn. Similarly mark the positions of P as the crate is tipped the 2nd, 3rd and 4th times.

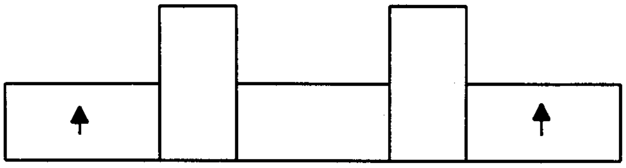
Next we will draw the locus of P.
When the crate is first tipped it turns about W. So P will follow an arc of a circle, centred on W, from its first position to its second position.

1 Put your compass point on W and draw an arc from the first to the second position of P. Next the crate rotates about X so put your compass point here and draw an arc from the second to the third positions. Complete the diagram to show the full locus of P.

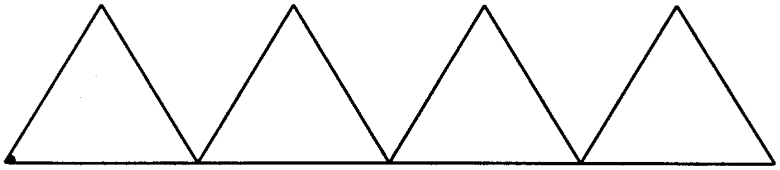
EXERCISE 2

Draw the following loci:

a **A crate whose end is twice as long as it is wide is moved across a floor by tipping it. Draw the locus of the top left corner of the crate if it is tipped until it is upright again. (See the diagram below.)**



b A triangular prism is similarly moved across a floor. Draw the locus of the marked corner.



c A goat is tethered by a rope 3m long. The other end of the rope is attached to a ring which slides along a wire fixed to two stakes 9m apart which are driven into the ground. Make a scale drawing showing the locus of the goat's end of the rope as the goat moves about always keeping the rope taut.

d A spider at the centre of a clock face crawls along the second hand at a constant speed. Draw a circle of radius 6cm on squared paper and the horizontal and vertical diameters. Draw also the two diagonal diameters by using the diagonals of the squares on your page. If it takes the spider 1 minute to reach the tip of the hand and the hand is initially pointing upwards plot 9 positions of the spider and draw in the locus of the spider's path.

THE CYCLOID

Suppose that a light is fixed to the rim of a bicycle tyre. What will be the locus of the light as the bicycle travels forward in a straight line?
Although the wheel is moving in a circle it is also moving forward and so the locus results from the combination of the two motions.

We can find the locus approximately by using a coin with a point marked on its edge. It will probably be easier if you work with a friend for this.

- Draw a horizontal line on a sheet of plain paper and mark a dot near the left-hand end.

Place a ruler along the line and put the coin so that the marked point is just touching the line and the ruler as shown above (with the mark on the coin at the mark on the line).

Now one person needs to hold the ruler still and firm while the other rolls the coin to the right.

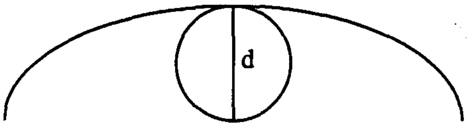
It is important that the coin rolls and does not slide and this is best done by moving it by a small amount at a time by using your finger at the very top of the coin. It may be best to practice this first.

When the marked point has moved about '/2 - 1 cm put a dot on the paper below the mark on the coin. Then roll the coin further and put another dot.
Continue until the mark on the coin is back on the base line.
If the coin slides at any time go back to the last dot and continue from there. Now you can draw a smooth curve through the dots.

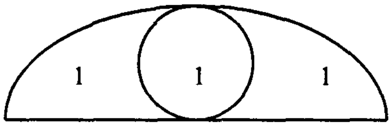
This locus is called a cycloid, it is the path of a point on the circumference of a rolling wheel. Write the name on your diagram.

The cycloid has many important and useful properties. For EXAMPLE:

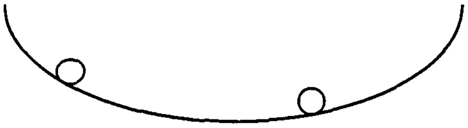
1. the length of the curve is 4 times the diameter of the circle;



2. the area is 3 times the area of the circle;



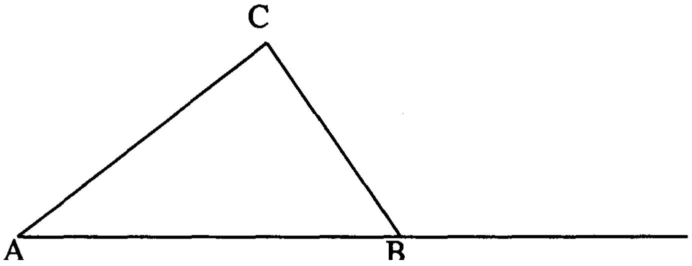
3. balls released from any height in a cycloid shaped ramp reach the bottom at the same time.



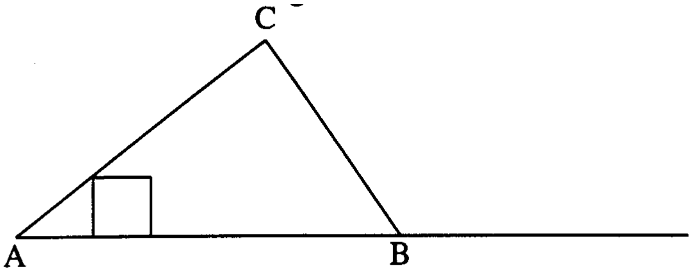
EXERCISE 3

Draw the following loci (use graph paper):

a Draw a triangle like the one shown below- it does not have to be the same shape or size, but make the base, AB, at least 10cm. Extend the base line to the right as shown.



From a point 1 cm to the right of A draw a vertical line up until AC is reached.
Measure the height of this line using the squares on the graph paper and use this measurement to complete the square which has this length for its left-hand side, as shown below.



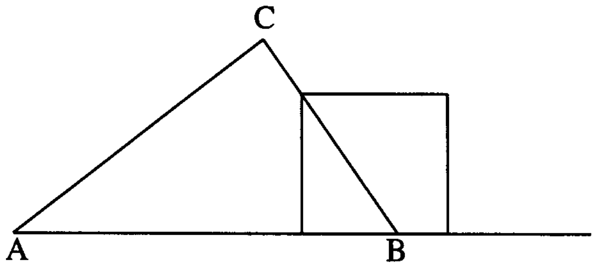
Next, starting at a point 2cm to the right of A, draw another line vertically up to AC and complete another square to the right of this line.

Imagine a series of squares, all with their base on the base line and their top left corner on AC. You have to construct the locus of the top right corner of the square as it moves to the right. Put a coloured dot at the top right corner of each of the squares you have drawn so far.

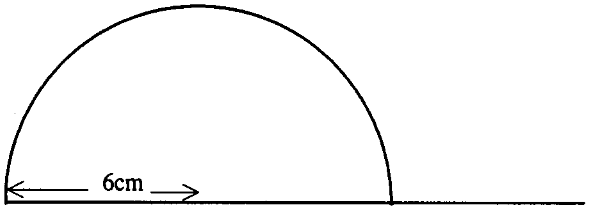
Draw in all the squares you can until C is reached. If you have not got a square with C as its top left corner, draw this in.

Now draw in the locus by drawing a line from the top right corner of the largest square through all your dots and ending at A.
If you imagine the smallest square becoming smaller as it moves to the left you will see that the locus gets closer and closer to A and that is why we draw the locus to the point A.

b Now suppose that as the square continues to move to the right the top left corner of the moving square moves along CB. Find the locus of the top right corner as the squares get smaller and smaller and the left corner goes down CB: see the diagram below.

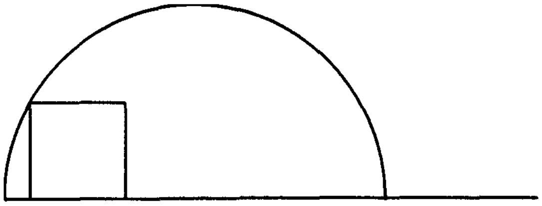


c Using your compasses draw a semicircle which has a radius of 6cm and draw an extended base line as shown below.



The locus you are going to draw is again the top right corner of a square which moves so that its base is always on the base line.
But this time the top left corner must always be on the semicircle, instead of on the triangle.

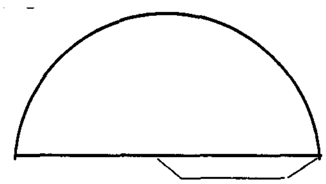
Draw a series of squares and plot the locus.
It will start at one end of the semicircle and end at the other end of the semicircle. One square is shown below.



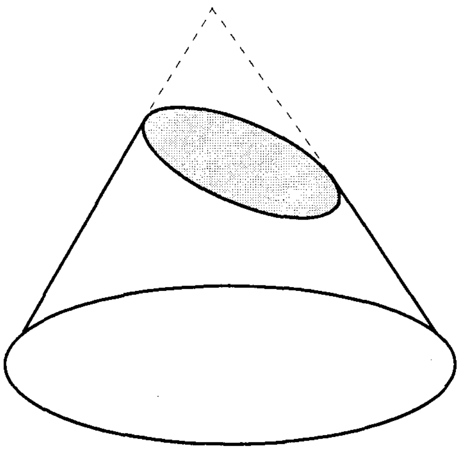
THE CONIC SECTIONS

The Conic Sections are certain curves which can be obtained by cutting a cone in different ways. These include the ellipse, the parabola and the hyperbola.

The locus for question (c) above, is in fact, part of an ellipse.
And although the orbits of the planets and their moons are approximately circular, the law of gravitation which they obey means they actually move in elliptical orbits.
To help visualise how the shapes are obtained from a cone you should draw a large semicircle with a radius of l Ocm on a sheet of plain paper.



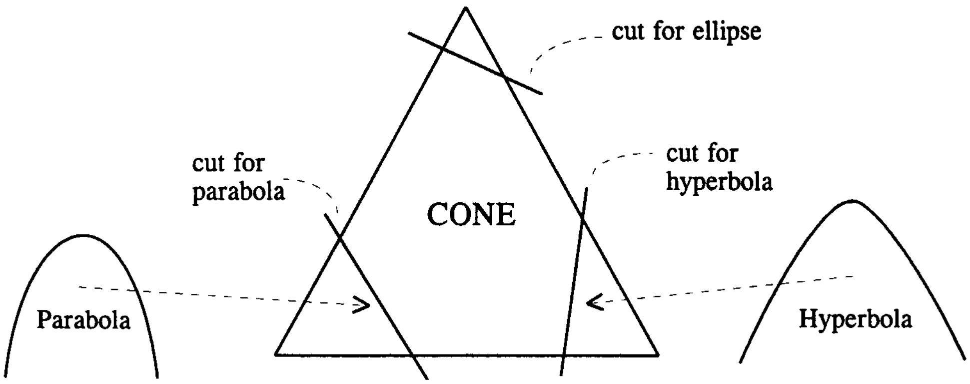
Cut out the shape, including the tab shown and glue one radius to the other to get your cone.



The diagram above shows how to cut a cone to reveal an elliptical face, or section.

If, instead, the cone is cut so that the cut is parallel to the opposite edge of the cone the resulting section is a parabola (see the diagram below).

Whereas if the cut is steeper than the cut for a parabola the result is a hyperbola.



The circle is also one of the conic sections.

2 How do you think the cone could be cut to reveal a circular section?

The conic sections can also be created in a different way as shown in the next EXERCISE.

We see ellipses very frequently. If you look around the room at something which you believe to be circular what you actually see is usually an ellipse. If a circle is viewed at an angle instead of from directly in front, the shape you see is an ellipse. So most circular shapes are actually seen as ellipses.

The parabola has already been discussed in an earlier chapter of this book. The trajectory of a projectile (unless it is thrown vertically up) is a parabola. The shape of the mirror in a car or bicycle head lamp is paraboloidal (a parabola rotated about its line of symmetry) and so are the mirrors used in reflecting telescopes.

The hyperbola is sometimes seen on a wall where a lamp casts a shadow of the top or bottom edge of the lampshade onto the wall.

All heavenly bodies: planets, comets, stars move in paths which are conic sections. They cannot move in any other way.

EXERCISE 4

You will need Worksheet 9 to draw some conic sections:

a The Worksheet shows a series of circles all centred on the same point. Mark this centre point with a colour and the letter P and note that the circles are numbered outwards from here. There is also a series of parallel lines. Draw over the line that goes through -6 in a dark colour.

We are going to draw the locus of a point which moves so that its distance from the line you have drawn over is the same as its distance from the point at the centre of the circles. The locus will be a parabola.

The most obvious point is the one at **-3** which is 3 units from the line and 3 units from P.
Mark this with a cross.
Now look for a point which is 4 units from both line and point. It will be on the circle marked with a 4. There are actually two of these, one above and one below the centre line.
Mark them both with a cross.
Next look for two points 5 units from both line and point and mark them.
Continue like this until you can go no further.
See if you can also draw in the points which are 3½ units from the line and P.
Now draw a smooth curve through the points and write **parabola** at the end of it.

b Next we will draw an ellipse.

We are looking for all points which are **twice as far from the line as they are from P**.
One point is at **-2** on the centre line as it is 2 units from the point and 4 units from the line.
Find the two points which are 3 units from P and 6 units from the line. We can write this as (6,3).
You can also plot a pair of points at (8,4) and (10,5) and a single point at (12,6).
It is also worth drawing a pair of points at (5,2½) and at (11,5½).
Join up the ellipse carefully and write its name at the end.

c Next we will draw a larger ellipse.

We now want all points which are **1½ times as far from the line as they are from P**.
You will find a pair of points at (4½,3), 3 units from P and 4½ units from the line.
Find as many points as you can which are 1½ times further from the line as they are from P.
A point you will probably not get but which you will need, however, is (3.6,2.4) which is on the centre line.
Draw and label the ellipse.

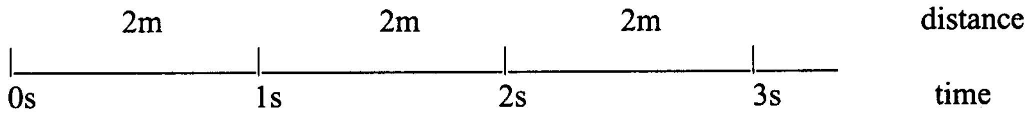
d A hyperbola can be drawn by looking for all points which are **twice as far from P as they are from the line**. One such point is (3,6).
Find as many points as you can and draw and label the curve.

36 Motion

We used the idea of movement in the last chapter and in this chapter we look at how motion can be quantified (described by numbers).
This involves bringing in the idea of time, with which you are already familiar.

SPEED

If an object, or mathematical point, moves steadily in a straight line so that it covers 2 metres in every second, we say its speed is 2 metres per second, abbreviated to 2m/s.



Speed is the distance covered in some unit of time.

EXAMPLE 1

So if a body travelled steadily and covered 50m in **5 seconds** we describe its speed by saying how far it goes in **1 second**.

In this case if it covers 50m in 5s then *proportionately* it will cover 10m in 1 second. The speed is therefore 10m/s (10 metres per second).

EXAMPLE 2

A train travels 100 miles in 2½ hours. What is its speed (assuming it is constant)?

We need to find out how far it travels in **1 hour**.
We do this by dividing 100 by 2½.
This gives 40 so the speed is 40 miles per hour or 40 mph.

These speeds are obtained by dividing the distance by the time:

$$\boxed{\textbf{Speed} = \frac{\textbf{distance}}{\textbf{time}}}$$

EXERCISE 1

Find the speed of the following, assuming it to be constant:

a A boat that covers 30 miles in 2 hours **b** A plane that covers 832 miles in 4 hours.

c A car that covers 50m in 4 seconds **d** A bicycle that travels 7 km in 3 hours.

e An object covers 10 miles in half an hour. What is its speed **i** in mph
ii in miles per minute.

f A walker covers 3 km in 2 hours. What is the speed in **i** km/h
ii metres per hour
iii metres per minute.

g Which represents a faster speed: 30 km covered in 4 hours or 50 m covered in ½ minute?

If in the last EXAMPLE, the speed of the train was not constant then our answer must say that the train travelled at an average speed of 40 mph.

$$\boxed{\textbf{Average speed} = \frac{\textbf{total distance}}{\textbf{time taken}}}$$

EXAMPLE 3

Find the average speed of an object that travels 10m in 3 seconds and then 20m in 7 seconds

The total distance travelled is 10 + 20 = 30m.
The total time taken is 3 + 7 = 10 seconds.
Therefore the average speed is 30 ÷ 10 = 3 m/s.

EXERCISE 2

Find the average speed for a body which travels:

a 50m in 10 seconds and then 150m in 40 seconds.

b 25 cm in 3.6 s and then 15 cm in 1.4 s.

c 60 cm in 8 s, then 60 cm in 2 s.

EXAMPLE 4

How far will an object moving at a constant speed of 15 m/s travel in 2 minutes?

The object travels 15 metres in every second.
So we just need to know how many seconds there are in 2 minutes.
This is clearly 120 so the object travels 15m 120 times.
$15 \times 120 = \underline{1800\text{m}}$.

We can also use the formula: speed = $\frac{\text{distance}}{\text{time}}$.
If we write this as $s = \frac{d}{t}$ we can rearrange to get $d = s \times t$ and $t = \frac{d}{s}$.

The formula d = s x t can be used when we need distance, as we do in the EXAMPLE above. The formula tells us to multiply speed by time. So we multiply the speed, 15, by 120 (we must change the 2 minutes to 120 seconds otherwise we would be mixing the units: seconds and minutes).

Similarly we can use $t = \frac{d}{s}$ to find time as the next example shows.

EXAMPLE 5

How long would it take a car travelling at a constant speed of 20m/s to travel 1km?

We change 1km to 1000m since the speed involves metres.
Then $t = \frac{d}{s} = \frac{1000}{20} = \underline{50 \text{ seconds}}$.

EXERCISE 3

How far would a body travel at:

a 50 mph for 2½ hours

b 24 m/s for 26 seconds

c 55 km/h for ½ hr

d 200 m/s for 3 minutes

e 3 cm/s for 3 hours

f 80 metres/min for 2 hr?

g Mr Jones sets off on a journey at 8.30 am and drives at a steady speed of 40 mph.
If he arrives at his destination at 9.50 am how far was his journey? ☞

How long will a journey take that covers:

h 40 m at 8 m/s

i 100 miles at 5 mph

j 80 km at 16 km/h

k 60 m at 100 m/s

l 3 miles at 20 mph

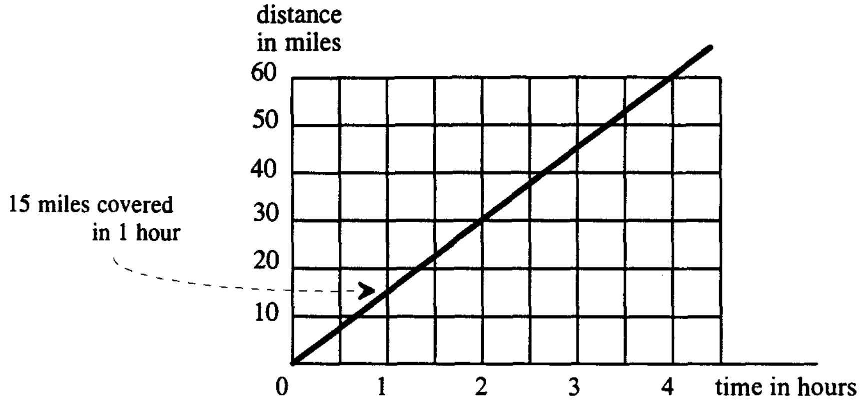
m 3 m at 4 cm/s

n An object travels 30m at 5m/s and then 35m at 7m/s. Find the time for each part of the journey and hence the time for the whole journey.

o A body travels at 50cm/s for 10 seconds and then at 20cm/s for 1 minute. Find the total distance travelled.

TRAVEL GRAPHS

A speed can be easily represented on a graph.



Look carefully at this graph.

Time is shown on the horizontal axis and distance on the vertical axis.
The bold line drawn represents the journey of a cyclist travelling at a speed of 15 miles per hour.
The line starts at the origin because she has travelled no distance in no time.
After 1 hour she has travelled 15 miles and you can see that the line goes through the point shown corresponding to 15 miles and 1 hour.

You will also see that the line goes through 30 miles in 2 hours, 45 miles in 3 hours and 60 miles in 4 hours.

So the line represents a journey, at a speed of 15 mph, for over 4 hours.

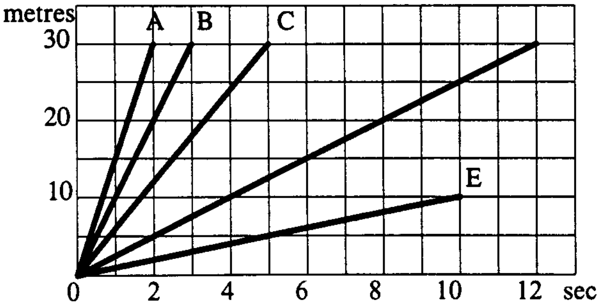
- Fold a sheet of graph paper in half and copy the graph above: use 2cm for 1 hour and 2cm for 10 miles. At the right-hand end of the line write 15 mph.

EXERCISE 4

On the same graph draw lines representing speeds of:

a 10 mph, **b** 5 mph, **c** 20 mph, labelling each line.

d Draw a line for a journey which covers 50 miles in 4 hours (write the speed at the end of the line).



The graph above shows five speeds.

e Which do think is the fastest speed, and which is the slowest speed?

f Find and write down the 5 speeds represented.

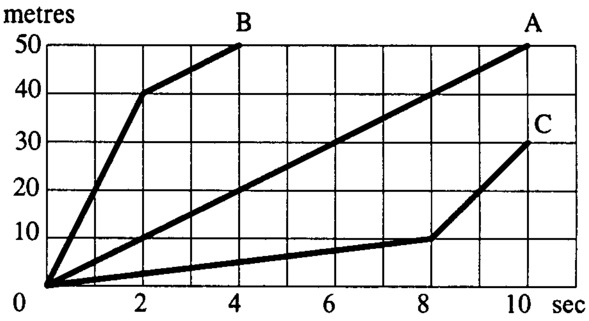
Reading directly from the graph, find **g** how far D goes in 8 seconds,

h how long it takes for B to travel 20m.

You should have found in the EXERCISE above that steeper lines represent faster speeds, less steep lines represent slower speeds.

CHANGE OF SPEED

The graph below shows three journeys: A, B and C.



1 A's journey has a constant speed. What is the speed in m/s?

C changes speed after 8 seconds.
In the first part of C's journey 10m are covered in 8 sec. So C's speed is $\frac{10}{8}$ = 1¼ m/s.
In the second part of C's journey 20m are covered in 2 seconds. So this speed is 10 m/s.
We can describe C's journey as: 1¼ m/s for 8 seconds followed by 10 m/s for 2 seconds.

2 Find the two speeds for B's journey and describe B's journey.

C covers 30m in 10 seconds overall. So we can say that C's average speed is 3 m/s.

3 Find B's average speed for the whole journey.

4 How far does C travel in the first 9 seconds of its journey? 5 How long does it take A to travel 30m?

EXERCISE 5

On your graph paper draw a time axis from 0 to 6 sec using 2cm for 1 sec and number the vertical axis from 0 to 120 with 2cm equal to 20 m.
Draw the following journeys:

a A travels at 10m/s for 6 seconds

b B travels at 40m/s for 2 seconds and then 5m/s for 4 seconds.

c C travels at 5m/s for 4 seconds and then 50m/s for 2 seconds.

d What is B's average speed and C's average speed?

e How far does B go in the first 4 seconds?

f How far does C go in the last 4 seconds of its journey?

CONVERSION GRAPHS

There are two main systems of units in use: the Metric system which is based on the number 10 and powers of 10 and the Imperial system which is much older than the Metric system.

So, for EXAMPLE, for measuring amounts of liquid we have litres in the Metric system (1 litre is equal to 1000m1) and we have pints in the Imperial system (8 pints equal 1 gallon).

Similarly we measure distance in kilometres or miles.

There is no exact relationship between these pairs of units and we sometimes need to convert from one unit to the other. We can use a conversion graph for this.

We will draw a conversion graph for converting a number of miles to a number of kilometres, or kilometres to miles.

6 On your graph paper, with the origin at the bottom left draw a horizontal axis from 0 to 10 with 1 cm for 1 unit. This is the miles axis: write miles at the end.

The vertical axis also has 1 cm for 1 unit and goes up to 16. Number this and write kilometres at the top.

Now, it is known that 5 miles are approximately the same as 8 kilometres.
So 10 miles will be equivalent to 16 kilometres on the graph.

- Find the point on your graph which is above 10 miles and on the same horizontal line as 16 kilometres. Join this point to the origin.

This is now a conversion graph because we can use the line to read off the equivalent of a number of miles in kilometres (or the other way around).

First note carefully what the small squares represent on the graph.
If you are using 2mm graph paper (where each of the smallest squares is 2mm wide) then there will be 5 squares to each mile, which means that each square represents $\frac{1}{5}$ or 0.2 of a mile.
The same happens on the other axis: each small square is 0.2km.

EXAMPLE 6

Suppose we want the equivalent of 7 miles.

We find 7 on the horizontal axis and follow the line vertically upwards until we reach the line.
Then we have to read off the value of that point on the vertical axis.
You should find it to be 1 small square above 11km, which makes it 11.2 km.

EXAMPLE 7

Convert 5.8 km to miles.

Find 5.8 on the km axis.
Go horizontally along from there until you reach the line.
Read off the value of that point on the miles axis.
You should arrive at 3.6 miles.

EXERCISE 6

Convert the following miles to kilometres:

a 9 **b** 2 **c** 6.4 **d** 0.8

Convert the following kilometres to miles:

e 7 **f** 15.2 **g** 8.2 **h** 4.3

7 25 litres are approximately equal to 44 pints.
Draw a conversion graph where 2cm = 10 pints on the horizontal axis and 2cm = 5 litres on the vertical axis.
Find the point where 25 litres = 44 pints and join this to the origin. Also extend this line in the other direction.

EXERCISE 7

Convert the following pints to litres:

a 30 **b** 45 **c** 21 **d** 7

Convert the following litres to pints:

e 20 **f** 30 **g** 11 **h** 14

Some other useful equivalents are: 2½cm ≈ 1 inch,
39 inches ≈ 1 metre,
1¾ pints ≈ 1 litre,
2.2 lb ≈ 1 kg.

37 Auxiliary Fractions

In this chapter we will extend the methods of finding recurring decimals in three directions. The Auxiliary Fractions, of which there are two types, are a great help in this.

AUXILIARY FRACTIONS FIRST TYPE

The first type involves replacing the denominator of a fraction by its Ekadhika.

EXAMPLE 1

For $\frac{17}{19}$ the Auxiliary Fraction (AF) is $\frac{1.7}{2}$.

The AF is found by replacing the denominator by its Ekadhika and dividing the numerator by 10.
So 19 is replaced by 2 and 17 is replaced by 1.7.

EXAMPLE 2

For $\frac{5}{28}$ the AF is $\frac{0.5}{3}$.

The 28 is replaced by *One More than the One Before*, which is 3, and the 5 is divided by 10 to give 0.5.

EXERCISE 1

Write down the Auxiliary Fraction for each of the following:

a $\frac{33}{49}$ **b** $\frac{15}{79}$ **c** $\frac{53}{88}$ **d** $\frac{4}{37}$ **e** $\frac{8}{19}$ **f** $\frac{22}{139}$ **g** $\frac{1}{19}$

The Auxiliary Fraction helps us to do the calculation: it helps to get us started and also gives us the Ekadhika, which we use continually.

EXAMPLE 3

Find the recurring decimal for $\frac{3}{7}$.

We know from Chapter 1 that we can do this by first obtaining a 9 in the denominator. So we first multiply by 7 to get $\frac{3}{7} = \frac{21}{49}$. The AF for $\frac{21}{49}$ is $\frac{2.1}{5}$.

We then simply divide 5 into 2.1 in the usual way, and keep dividing by 5.
5 into 2.1 goes 0.4 with remainder 1 which is put before the 4:

$$AF = \frac{2.1}{5} \text{ therefore } \frac{3}{7} = 0._{1}4$$

Continuing we get $\frac{3}{7} = 0._{1}\dot{4}_{4}\,2_{2}8\,5\,7\,\dot{1}$

Note that the answer is $0.\dot{4}\,2857\,\dot{1}$, the remainders are not part of the answer.

This is just the same as in Chapter 1 but with the additional help of the Auxiliary Fraction to get us started.

EXERCISE 2

Use Auxiliary Fractions to find recurring decimals for:

a $\frac{14}{39}$ **b** $\frac{6}{7}$ **c** $\frac{1}{13}$ **d** $\frac{21}{23}$

DENOMINATORS ENDING IN 8, 7, 6

If a denominator ends not with a 9 but with a number close to it (8, 7 or 6) we can use the same method as for denominators ending in 9 but multiply the last figure at each step by 2, 3 or 4 before dividing.

EXAMPLE 4

$$\frac{7}{48} = 0._{2}1_{2}4_{3}5\,8_{1}\,\dot{3}$$

The AF is $\frac{0.7}{5}$ (because 48 is still close to 50) so we begin by dividing 5 into 0.7.
This gives us 0.1 remainder 2: $\frac{7}{48} = 0._{2}1$

So far this is just what we would have done for $\frac{7}{49}$. But from now on, before we divide by 5 we must **double the last figure**, because 48 is **2** below 50.

☞

Looking at ${}_21$ we double the last figure and therefore divide 5 into 22 rather than 21. This gives 4 remainder 2, written ${}_24$ so we now have: $\frac{7}{48} = 0.{}_21{}_24$

We again double the last figure and divide 5 into 28 to get ${}_35$. Then doubling the 5 in 35 we divide 5 into 40, and so on.

- Check that you agree with the whole answer above.

Note how the 3 repeats itself at the end: we keep dividing 5 into 16 and get 13 over and over again.

We can summarise the whole process as follows:

1. Find the Auxiliary Fraction. E.g. $\frac{0.7}{5}$ in the example above.
2. Divide the bottom of the AF into the top, writing any remainder **before** the answer figure. E.g. $0.{}_21$
3. From now on **double** the last figure of the number pair before dividing. E.g. 5 into 22 (not 21).

EXERCISE 3

Find the full recurring decimal for:

a $\frac{5}{38}$ **b** $\frac{5}{78}$ **c** $\frac{7}{18}$

d Find the first 5 figures (after the decimal point) of the decimal for $\frac{13}{28}$.

EXAMPLE 5

$$\frac{78}{87} = 0.{}_6\dot{8}{}_39{}_36{}_35\;5{}_61\;7{}_32\;4{}_31{}_63{}_67\;9\;3\;1{}_30{}_33{}_34{}_64\;8{}_62{}_37{}_65{}_38\;6\;2{}_60{}_6\dot{6}$$

The AF is $\frac{7.8}{9}$ so we start with 9 into 7.8 goes 0.8 remainder 6 giving: $0.{}_68$

We now see 68 but because 87 is 3 below 90 we must treble the last figure at each step. So the 8 in 68 becomes 24, and added to 60 makes 84.

Dividing 84 by 9 we get 9 remainder 3 giving: $0.{}_68{}_39$

We now have 39. We treble the 9 to 27 and add the 30 which gives 57. Then 57 divided by 9 goes 6 remainder 3 giving $0.{}_68{}_39{}_36$ and so on.

- Check you agree with the answer above.

EXERCISE 4

Find the full recurring decimal for:

a $\frac{9}{37}$ **b** $\frac{34}{77}$. **c** Find the first 6 figures of the decimal for $\frac{22}{57}$

1 Find $\frac{9}{28}$ giving 7 figures after the decimal point.

In finding $\frac{9}{28}$ you should have arrived at ${}_{1}5$ in the 7th decimal place.

If you continue from here you get 3 into 20 (because the 5 is doubled) which goes 6 remainder 2.

This gives us ${}_{1}5{}_{2}6$.
Then 3 into 32 gives a 2-figure answer (10 rem 2).

This means that the 6 in ${}_{2}6$ needs to be a 7 because 1 will be carried back from the next place.

So we do not write 3 into 20 goes 6 remainder 2 but write 7 remainder $\bar{1}$.

So we now have ${}_{1}5{}_{\bar{1}}7$.

This works nicely because we double 7 to get 14 and add this to $\overline{10}$ to get 4.
Then 3 into 4 goes 1 remainder 1 and the decimal is starting to repeat.

This gives $0.32{}_{1}\dot{1}\,4{}_{2}28{}_{1}5{}_{\bar{1}}\dot{7}$ as the full answer.

EXERCISE 5

Find the first 9 figures of the decimal for:

a $\frac{20}{47}$ **b** $\frac{19}{67}$

AUXILIARY FRACTIONS- SECOND TYPE

So far all the denominators we have encountered have been just below a multiple of ten, like 29, 38, 47 etc.

The second type of auxiliary fractions helps us to handle fractions which are just over a multiple of ten.

> To get the Auxiliary Fraction for fractions in which the denominator ends in a 1
> the numerator and denominator are both reduced by 1
> and the top and bottom are divided by 10.

EXAMPLE 6

For $\frac{7}{31}$ the AF is $\frac{0.6}{3}$.

Reducing top and bottom by 1 gives $\frac{6}{30}$ then 6 and 30 are divided by 10.

EXAMPLE 7

For $\frac{1}{21}$ the AF is $\frac{0.0}{2}$.

EXERCISE 6

Write down the Auxiliary Fraction for:

a $\frac{9}{41}$ **b** $\frac{3}{91}$ **c** $\frac{44}{71}$ **d** $\frac{1}{61}$ **e** $\frac{2}{11}$ **f** $\frac{32}{301}$

The Auxiliary Fraction, as before, helps to get us started with the decimal.
In fact the method is the same as before but with one important difference: before dividing at each step we take the last figure of the number being divided from 9.

EXAMPLE 8

$$\frac{19}{31} = 0.6\ 1{}_{2}2\ 9\ 0\ 3\ 2{}_{1}2\ .\ .\ .$$

The AF is $\frac{1.8}{3}$ and so we begin by dividing 3 into 18. This gives: 0.6

From now on we take each answer figure (but not any of the small remainder figures) from 9 before dividing.

So we take 6 from 9 to get 3. Then 3 divided by 3 gives **1**: 0.61

Take this 1 from 9 to get 8, 8÷3=**2** rem **2**, so we now have: $0.61{}_{2}2$

Take the last 2 in 22 from 9 to get 27, 27÷3=**9**: $0.61{}_{2}29$

Take 9 from 9 to get 0, 0÷3=**0**: $0.61{}_{2}290$

Take 0 from 9 to get 9, 9÷3=**3**, and so on.

2 Check you agree with the remaining figures shown above and continue the division. The decimal begins to recur after 15 figures.

To summarise: 1. Write down the Auxiliary Fraction.
2. Divide the bottom of the AF into the top and put down the result, prefixing any remainder.
3. Take the last figure of the last number pair from 9 and divide again.

EXERCISE 7

Find the full recurring decimal for:

a $\frac{13}{21}$ **b** $\frac{22}{31}$ **c** $\frac{1}{21}$ **d** $\frac{17}{41}$ **e** $\frac{8}{51}$ **f** $\frac{4}{91}$ **g** $\frac{1}{81}$

WORKING 2,3 ETC FIGURES AT A TIME

It is possible to speed up the calculation even further by working with groups of 2, 3, 4, 5, 6 or more figures at a time.

EXAMPLE 9

$$\frac{1}{199} = 0.{}_{1}0\ 0/5\ 0/2\ 5/{}_{1}1\ 2/5\ 6/2\ 8/1\ 4/0\ 7/{}_{1}0\ 3\ \ldots.$$

The AF is now $\frac{1}{2}$ but because we have two 9's in the denominator we can get the answer **two figures at a time**.

We begin from the AF with 2 into 1 goes **00** remainder **1**, so we put down

$$0.{}_{1}00$$

This gives 100, and 2 into 100 goes **50**, which we put down: $0.{}_{1}00/50$

(Note each pair of answer figures are separated by an oblique line only to help show the method)

Then 2 into 50 is **25**: $0.{}_{1}00/50/25$

Then 2 into 25 is **12** remainder **1**, so we put ${}_{1}12$: $0.{}_{1}00/50/25/{}_{1}12$

Then 2 into 112 and so on.

We get the answer at least as fast as we can write it down.

3 Check the answer above and continue for at least six more pairs of digits.

EXAMPLE 10

$$\frac{59}{133} = \frac{177}{399} = 0.{}_{1}4\ 4/3\ 6/0\ 9/{}_{1}0\ 2/{}_{2}2\ 5/{}_{1}5\ 6\ \ldots.$$

Here we see 133 in the denominator and so we multiply the top and bottom of $\frac{59}{133}$ by 3 to get $\frac{177}{399}$ to get 9's in the denominator.

So the AF is $\frac{1.77}{4}$ and we therefore divide by 4, two figures at a time.

4 into 177 goes 44 remainder 1 etc.

EXAMPLE 11

$\frac{108}{2001} = 0.{}_{1}0\ 5\ 3/9\ 7\ 3/0\ 1\ 3/4\ 9\ 3 \ldots.$

AF = $\frac{0.107}{2}$ (Note that we are using the second type of AF here so we reduce the numerator of the given fraction by 1) and we work with groups of 3 figures:

107 ÷ 2 = **53** remainder **1** so we put down: $0.{}_{1}053$

Next take each of the figures 053 from 9 to get 946, so 2 into 1946 = **973**:

$0.{}_{1}053/973$

Then 973 becomes 026, and dividing by 2 we get **013**.
Etc.

EXERCISE 8

Find the first 5 groups of digits in the following:

a $\frac{88}{199}$ **b** $\frac{45}{299}$ **c** $\frac{535}{4999}$ (groups of 3 here)

d $\frac{70}{233}$ (multiply by 3 first) **e** $\frac{57}{201}$ (groups of 2 and take from 9 after the first group)

f $\frac{19}{43}$ (multiply by 7 first) **g** $\frac{222}{19999}$

h $\frac{37}{198}$ (divide by 2 in groups of 2 and double the last 2 figures after the first step)

The special numbers which we used in an earlier chapter can be very useful here.
For example $\frac{1}{67}$ can be written as $\frac{3}{201}$ giving the answer 2 figures at a time,
or alternatively as $\frac{597}{39999}$ giving the answer 4 figures at a time.

38 Surveys

In this chapter we will extend our earlier work on "Charts". We will learn other ways of displaying data, how to construct questionnaires for surveys and how to construct scatter diagrams. You will be asked to carry out a survey on something that interests you, involving a questionnaire.

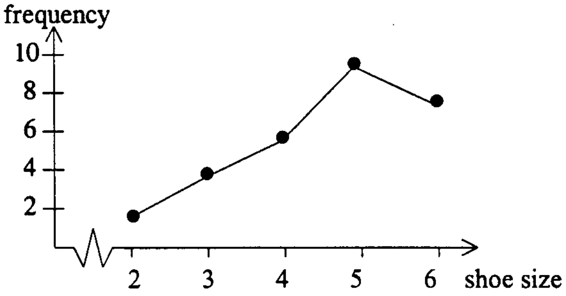
FREQUENCY POLYGONS

A type of chart not considered before is the Frequency Polygon.
Consider the results of the shoe size survey from the chapter on Charts.

Shoe Size	**Frequency**
2	2
3	4
4	6
5	10
6	8
	TOTAL=30

Instead of drawing a dot diagram, line chart or bar chart, we could draw a frequency polygon by putting a dot on the chart above the relevant shoe size at a height which corresponds to the frequency. Then the dots are joined with straight lines as shown below.

Frequency Polygon



The horizontal scale here has a jagged line near the origin. This is to indicate that the first number, 2, is not actually 2 units from the origin. You will need to use this in the next question.

1 Copy out and complete the following frequency table and then draw a frequency polygon for the data.

A class of children measured their hand spans to the nearest cm. Here are the results:

Handspan in cm	Tally	Frequency
9	||	2
10	𝍸 |	6
11	𝍸 𝍸 ||	
12	𝍸 𝍸	
13	𝍸	
14	|	
		Total =

2 If you have still got some data which you collected from the earlier Charts chapter construct a frequency polygon from it.

PIE CHARTS

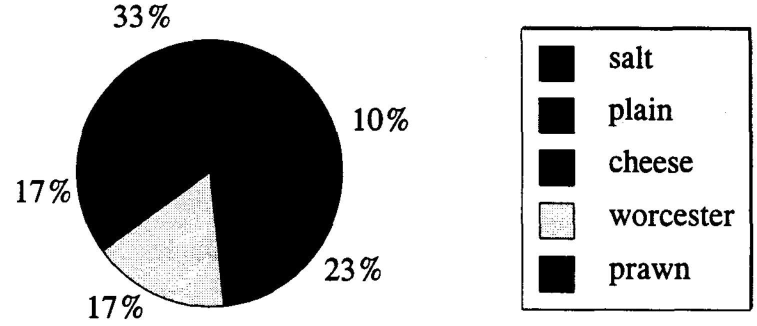
Pie charts are a convenient way of presenting data in a visually attractive form.
The results of a survey can then be seen easily *By Mere Observation.*
There are three main ways of constructing pie charts.

(i) Using a Computer

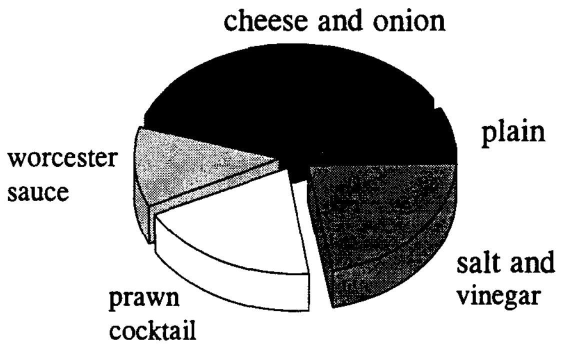
All of the charts we have considered so far can be drawn using a computer. One simple way to do this is to use a spreadsheet. If you are not familiar with this process ask your teacher to show you. You are normally given a wide choice as to the way you can present your results. With a pie chart you can draw them flat or in 3-dimensions, have segments separated or joined and have the percentages marked on the diagram or not indicated. Two examples are shown below for the results of a survey on favourite crisps.

Flavour of Crisp	Frequency
Plain	3
Cheese and Onion	10
Worcester Sauce	5
Prawn Cocktail	5
Salt and Vinegar	7
	TOTAL = 30

Favourite Crisps



Favourite Crisps



<u>(ii) Using a Pie Chart Scale</u>

We first need to convert the frequencies into percentages.
We then use a pie chart scale to construct the pie chart.

Since there were 3 people out of 30 who preferred plain crisps we convert this to a percentage in the usual way: $\frac{3}{30}\times100 = 10\%$.
Similarly for the others: $\frac{10}{30}\times 100 = 33\frac{1}{3}\%$, $\frac{5}{30}\times 100 = 16\frac{2}{3}\%$, $\frac{7}{30}\times 100 = 23\frac{1}{3}\%$..

Finally check that the percentages add up to 100: $10 + 33\frac{1}{3} + 16\frac{2}{3} + 16\frac{2}{3} + 23\frac{1}{3} = 100$.

For drawing the chart we can round these percentages to the nearest percent:
10%, 33%, 17%, 17%, 23%.
(Note however that the numbers may not now add up to exactly 100).

You can then draw around the pie chart scale marking off the correct percentages around the circle.

(iii) Using a Protractor

Since a full circle contains 360°, to find the angle corresponding to the frequency of 3 for plain crisps, we know that 3 is $\frac{1}{10}$ of the total, 30, so we find $\frac{1}{10}$ of 360° which is **36°**.
Similarly we can see that 10 is $\frac{1}{3}$ of 30 so we find $\frac{1}{3}$ of 360° which is **120°**.
And 5 is $\frac{1}{6}$ of 30, so $\frac{1}{6}$ of 360° gives **60°**.
Finally 7 is $\frac{7}{30}$ of 30, $\frac{7}{30}$ of 360 = **84°**.

So the angles of the sectors are 36°, 120°, 60°, 60°, 84°.

These angles can then be drawn using a protractor.

3 Represent the shoe sizes shown at the beginning of this chapter in the form of a pie chart, using a pie chart scale.

4 Represent the hand spans data given earlier in the form of a pie chart, using a protractor.

GROUPING DATA

When data can only take certain definite values it is called discrete data.
For example numbers of people are always whole numbers and shoe sizes are of certain sizes. But other types of data, such as heights and weights do not take specific values and are called continuous data.

It is often useful with discrete or continuous data to put it into groups.
If some data had a very large range you would not want to draw a bar chart with, say, 100 bars. In this case you may want to group every 5 or so values together.

Suppose you recorded the following times (in minutes) for a particular car journey:
34, 27, 43, 42, 31, 22, 33, 53, 40, 47, 32, 41, 44, 50, 38, 48, 30, 28, 33, 26, 43, 26, 37, 36, 25, 31, 32, 44, 38, 39.

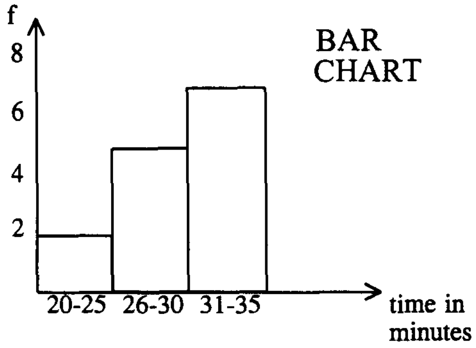
You may decide to construct a tally chart that groups the times in 7 groups:

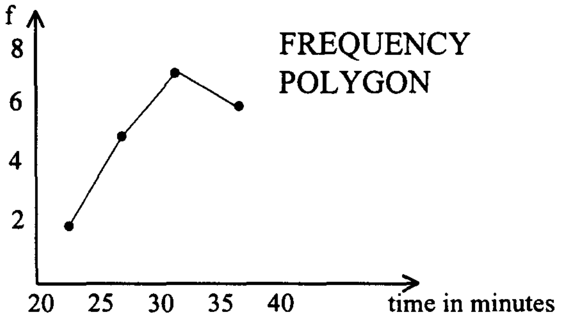
FREQUENCY TABLE

Time in minutes	Tally	Freq.
20 - 25	||	2
26 - 30	𝍸	5
31 - 35	𝍸 ||	
36 - 40	𝍸 |	
41 - 45		
46 - 50		
51 - 55		

TOTAL =

5 Copy and complete this frequency table.
Copy and complete the bar chart started below.
Copy and complete the frequency polygon started below. Use the midpoint of the interval, so for example, as the first interval is from 20 to 25 the first point goes above 22½.





6 The number of points a rugby team scores over 40 games in a season are as follows:
12, 5, 0, 8, 16, 0, 4, 7, 11, 24, 15, 3, 6, 0, 0, 18, 6, 15, 13, 12, 19, 7, 4, 8, 4, 12, 16, 22, 28, 14, 32, 16, 9, 15, 16, 5, 4, 3, 2, 12.

Group the data into appropriate and equal class intervals.
Draw a frequency table and a pie chart.

7 Conduct a survey of your own in your class which requires the data to be grouped. Draw an appropriate frequency table and graphs.

DESIGNING A QUESTIONNAIRE

Questionnaires are usually of two types. In one, the person answering the questions fills in a sheet with questions on it. In the other type the person doing the survey asks the question verbally and records the answer.

When designing a questionnaire the following points should be considered:
● keep the question short and simple,
● do not ask leading questions (such as "don't you agree that fox hunting should be banned?"),
● if it is a written questionnaire make it easy to follow. Leaving boxes to tick is a good method. The boxes could be labelled "yes" or "no" or you could have a series of boxes labelled "agree strongly", "agree", "neutral", "disagree", "disagree strongly".

A great deal of thought has to be put into designing a questionnaire. For EXAMPLE where are you going to ask for opinions? You may get a biased view if you asked only people in Manchester about their favourite football team.

The time at which the survey is carried out is also important. In carrying out a survey on the number of people in a car, for example there are likely to be more people in the car at weekends. You will also want to decide on the number of people to question.

The important thing is that the sample you interview are representative, that they represent the whole population of the town or country being studied and is as unbiased as possible.

8 Carry out a survey involving the use of a questionnaire.
Be quite clear exactly what it is that you are investigating.
Study your results carefully and present the results in some attractive ways.
Think about what the survey has shown you and write down your conclusions and inferences.

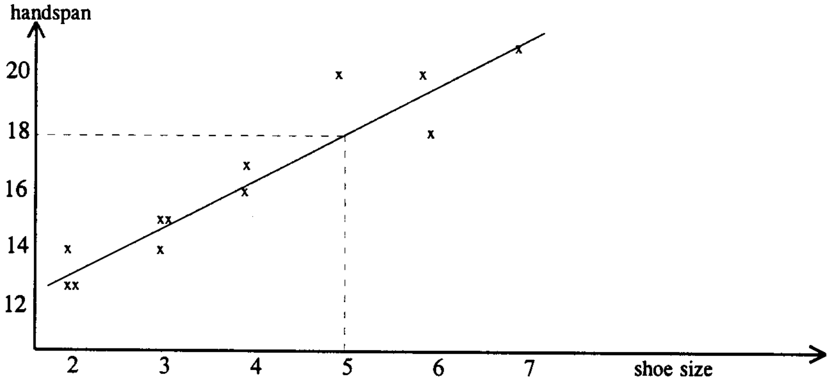
SCATTER DIAGRAMS

Sometimes we get two sets of data which correspond. For example we may have the weights and heights of a group of people.
A useful way of representing such data is on a scatter diagram.

The table below shows the shoe sizes and hand spans (to the nearest cm) of a group of 12 children.

shoe size	3	2	7	6	4	3	2	5	6	2	3	4
hand span	15	13	21	18	17	15	13	20	20	14	14	16

The scatter diagram for this is shown below.



The horizontal scale is used for the shoe sizes and the vertical scale is used for the hand spans. Then for each person a cross can be drawn at the correct point. For example the first person has a shoe size of 3 and a hand span of 15 so we put a cross above the 3 on the horizontal scale and at a height corresponding to 15 for the hand span. If we want to plot a point where one has already been drawn, draw them separately but close together.

If the crosses seem to lie roughly on a line, as they do in this example, we can draw a line through the crosses. We judge by eye where the best line goes and it usually has the same number of points above it as below. This is called the line of best fit.

Another important point about this diagram is that it shows that as shoe sizes increase hand spans increase also.

It is now possible to make predictions from this line of best fit.
If, for example, we wanted to have some idea of the hand span of a child with a shoe size of 5 we could use the diagram: we draw a line up from 5 on the horizontal scale, as shown dashed above, then read off the answer on the vertical scale. This indicates a hand span of 18cm.

9 a Use the diagram the estimate the hand span of a child with a shoe size of 6½.
b What shoe size would you expect for a child with a hand span of 14cm?
c Comment on your answers to **a** and **b**.

EXERCISE 1

Draw scatter diagrams for the following three sets of data. Comment on any pattern you find and draw a line of best fit if it is appropriate.

a A particular brand of car currently sells at £10,000 when new. 15 of these cars, but with different ages, were found to be on sale in a magazine. Their ages and cost are shown below:

age in years	2	4	3	2	5	7	6	8	4	1	3	8	3	7	6
cost in £	8,000	6,000	7,750	8,500	5,500	3,500	4,500	2,750	6,500	9,000	7,250	2,000	7,500	3,000	4,000

b Ten people in a company were asked their heights and salary. Here are the results:

height in cm	171	152	178	163	145	182	185	191	175	162
salary in £	28,000	14,000	20,000	22,000	20,000	12,000	25,000	15,000	12,000	15,000

c A chain of supermarkets examined its shops to see if there was a link between the number of customers per hour and the number of people living within a 5 mile radius:

no. customers per hour	1,500	1,200	2,000	1,000	600	800
no. people in 1000's in a 5 mile radius	60	55	80	30	20	10

d What reason can you give for the difference in the price of cars of different ages in a?

e What reason can you give for the difference in the price of cars of the same age?

f Is it possible to draw in a line of best fit for b above? If not, why not? Is this what you would expect?

g Using your line of best fit in **a** estimate **i** the cost of a car 4½ years old,
ii the age of a car costing £3,5000.

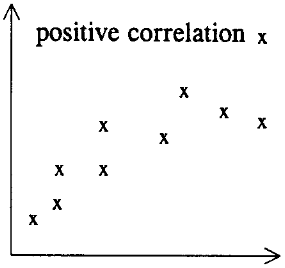
h At what age would the cars in **a** above cost £1,000? (you may need to extend your line of best fit).

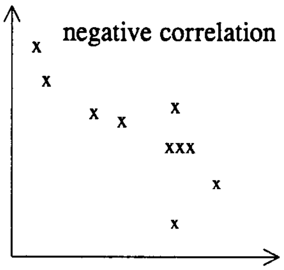
i In **c** above estimate the number of customers you would expect if there were 70,000 people within a 5 mile radius.

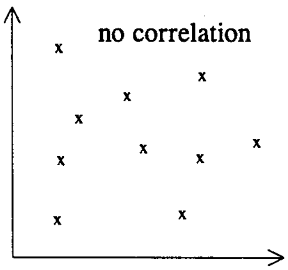
j **If a shop had 2,500 customers per hour how many people would you expect to be living within 5 miles?**

CORRELATION

If we think there is a link between two sets of data, as with the shoe sizes and hand spans, which is shown by the line of best fit we say there is correlation. If the line of best fit goes up to the right we call this positive correlation and if it goes down to the right it is called negative correlation. The three possible types of graph are shown below.







c above is an example of positive correlation: the number of customers per hour increases as the population within a 5 mile radius increases,
a is an example of negative correlation: the price of a car decreases as the age increases,
b is an example of no correlation in which there is no obvious relationship between the sets of data.

CODES

Sometimes you may want to send a secret message to someone.
But since there is always the possibility of the message being intercepted you may want to use a code so that no one except the intended receiver can understand it.

You could just replace every letter with a number corresponding to its position in the alphabet.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26

But this would be easy to decode.

You could change the word message into numbers as described above, add a number, say 3, to every number and then convert back to a letter. So if we want to code SMILE we get:

message	S	M	I	L	E	
	19	13	9	12	5	
	3	3	3	3	3	+
	22	16	12	15	8	
	V	P	L	O	H	message sent.

But this would also be quite easy to decode. The 3 is the key to coding and decoding this message.

A very good code is one similar to the one above, but where the numbers added on (3's in the example above) are different.

Suppose you and your friend agree to use the recurring decimal for $\frac{15}{29}$ as a key.

$\frac{15}{29} = 0.5\ 1\ 7\ 2\ 4\ 1\ 3\ 7\ 9\ 3\ 1\ 0\ 3\ 4\ 4\ 8\ 2\ 7\ \ldots$

Instead of adding 3's we add these numbers starting at the beginning:

message	S	M	I	L	E	
	19	13	9	12	5	
	5	1	7	2	4	+ numbers from recurring decimal
	24	14	16	14	9	total
	X	N	P	N	I	message sent

To make messages easy to read we can suppose that a space between words is given the number zero.

So MUST MEET would become:

M	U	S	T		M	E	E	T	
13	21	19	20	0	13	5	5	20	
5	1	7	2	4	1	3	7	9	+
18	22	26	22	4	14	8	12	29	
R	V	Z	V	D	N	H	L	C	message sent

To convert 29 to a letter we first subtract 26 (this is because there are 26 letters in the alphabet). So 29-26=3 and 3 becomes C.

1) Encode the following a) CODE b) SECRET c) I AM LOST

2) Encode SEND HELP QUICKLY using the recurring decimal for $\frac{7}{19}$ as a key.

Decoding a message is the reverse of encoding it.
The intended receiver of the message knows the key and subtracts the numbers given by the agreed recurring decimal before converting back to letters.

So to decode MFSRDNH (using the $\frac{15}{29}$ key again):

M	**F**	**S**	**R**	**D**	**N**	**H**		
13	**6**	**19**	**18**	**4**	**14**	**8**		convert to numbers
5	**1**	**7**	**2**	**4**	**1**	**3**	**-**	subtract key
8	**5**	**12**	**16**	**0**	**13**	**5**		
H	**E**	**L**	**P**		**M**	**E**		message

And the message RVZVDNHLC:

R	**V**	**Z**	**V**	**D**	**N**	**H**	**L**	**C**	
18	**22**	**26**	**22**	**4**	**14**	**8**	**12**	**3**	
5	**1**	**7**	**2**	**4**	**1**	**3**	**7**	**9**	**-**
13	**21**	**19**	**20**	**0**	**13**	**5**	**5**	**20**	
M	**U**	**S**	**T**		**M**	**E**	**E**	**T**	

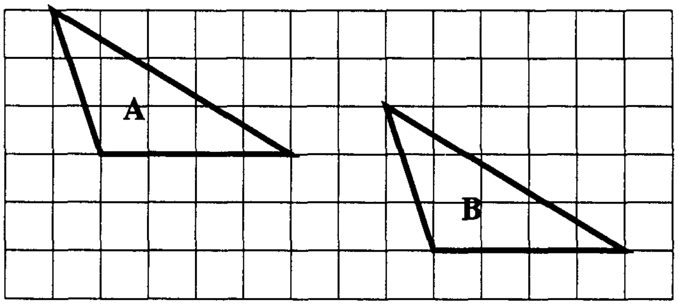
Here we find we need to take 9 from 3. When this happens we simply add 26 to the first number: so 3+26 = 29 and 29-9=20.

3) Decode the following (the key is still $\frac{15}{29}$): **a)** BJZJ **b)** XXHNPPZ
c) XBCGDNH d) RBUADTSPNV

4) Decode using $\frac{6}{13}$ as the key (remember recurring decimals repeat themselves indefinitely):

ERXFBADYQJOTDQPWUMGZMD

39 Vectors

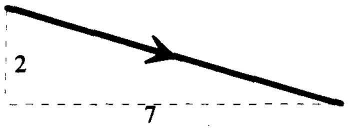


You may recall that we describe a **translation**, such as the one shown above from A to B, by using square brackets.

The translation above would be described by $\begin{bmatrix}7\\-2\end{bmatrix}$ because shape A is moved 7 units to the right and 2 units down.

Similarly a translation from B to A would be $\begin{bmatrix}-7\\2\end{bmatrix}$.

Now if we consider only the translation itself, without regard to the object being translated, we can represent this by a single arrow which goes 7 units to the right and 2 units down:



This arrow represents the translation, just as |??! 7does.

Anything that can be described by an arrow or a pair of numbers as shown above is called a vector.

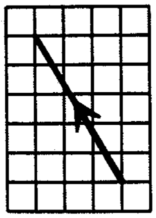
So a translation, or a shift from one place to another, is a vector.

When a pair of numbers is used it is called a column vector.

<u>EXAMPLE 1</u>

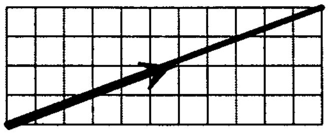
Draw an arrow for the column vector $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$.

We put a dot at some convenient point on a square grid. Then we go 3 units to the left (because the 3 is negative) and 5 units up and put another dot. Then we join the first dot to the second and put an arrow on the line pointing towards the second dot:



EXAMPLE 2

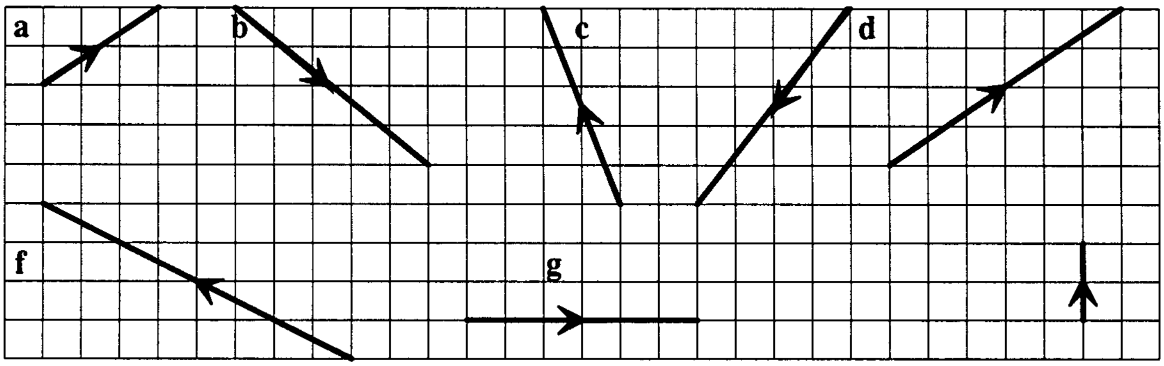
Write the vector below as a column vector.



The arrow goes 11 units in the positive x-direction and 4 units in the positive y-direction. So the answer is $\begin{bmatrix} 11 \\ 4 \end{bmatrix}$.

EXERCISE 1

Write the following as column vectors:



On a sheet of squared paper draw the following vectors:

i $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ **j** $\begin{bmatrix} 6 \\ -3 \end{bmatrix}$ **k** $\begin{bmatrix} -4 \\ -1 \end{bmatrix}$ **l** $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ **m** $\begin{bmatrix} 3\frac{1}{2} \\ -6 \end{bmatrix}$ **n** $\begin{bmatrix} -9 \\ -\frac{1}{2} \end{bmatrix}$

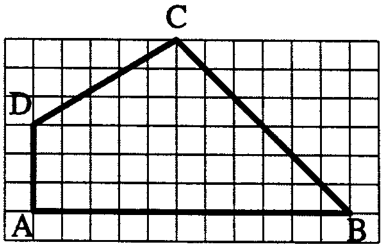
o $\begin{bmatrix} 0 \\ -3 \end{bmatrix}$ **p** $\begin{bmatrix} -4 \\ 0 \end{bmatrix}$

ANOTHER NOTATION

Vectors can also be described by referring to a letter at each end.

EXAMPLE 3

Describe the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{DB}$.



There are no arrows on the diagram but the direction is given by $\overrightarrow{AB}$ in which the arrow shows that the vector goes from A to B (the arrow always points to the right).

So $\overrightarrow{AB}$= $\begin{bmatrix} 11 \\ 0 \end{bmatrix}$.

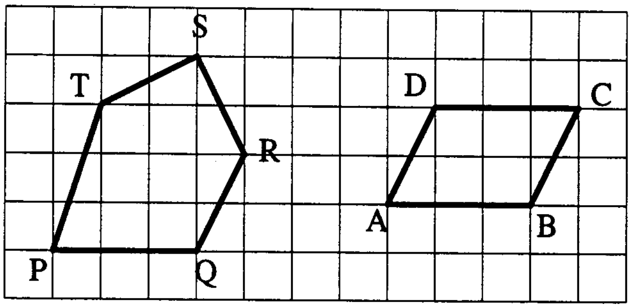
Similarly $\overrightarrow{BC}$ is the vector that goes from B to C. So $\overrightarrow{BC}$ = $\begin{bmatrix} -6 \\ 6 \end{bmatrix}$.

DB goes from D to B so $\overrightarrow{DB}$ = $\begin{bmatrix} 11 \\ -3 \end{bmatrix}$.

1 Write down column vectors for $\overrightarrow{BD}, \overrightarrow{AC}, \overrightarrow{CD}, \overrightarrow{AD}$.

EXERCISE 2

The following vectors can all be found in the diagram below. There is more than one answer to some of the questions.



Describe, using the letters at the end points, the vectors:

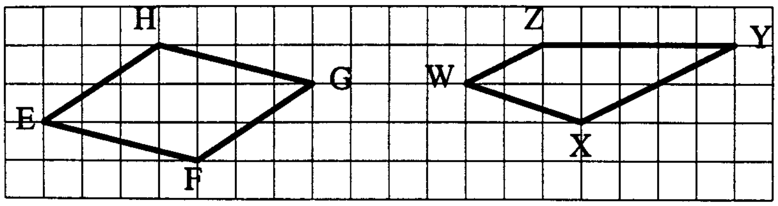
a $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ **c** $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **d** $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ **e** $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ **f** $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

The following vectors can also be described using the letters in the diagram above, although the lines are not drawn in. Describe the vectors, using letters, as above.

g $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ **h** $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ **i** $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ **j** $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$

Using the diagram below, write column vectors for:

k EH, FG, YZ, WZ, XY.



In question **k** above you should have found that $\overrightarrow{EH}$ and $\overrightarrow{FG}$ give the same answer.
We can therefore say that $\overrightarrow{EH} = \overrightarrow{FG}$. The vectors are equal.
Equal vectors have the same direction, which means that they are parallel, and they have the same length: the distance EH is the same as the distance FG.

If vectors are **equal** they are parallel and of equal length.

Also in the diagram for question k you can see that XY is parallel to WZ and has exactly twice the length.

- Look at the diagram and check you agree with this.

The column vectors were $\overrightarrow{WZ} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{XY} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

And here we can see that the numbers in the vector for XY are double those of WZ.

We could write $\overrightarrow{XY} = 2\overrightarrow{WZ}$.

Note that if a vector is multiplied by a number only its length is changed, not its direction.

EXAMPLE 4

If $\overrightarrow{AB} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ write down vectors for $3\overrightarrow{AB}$ and $\frac{1}{2}\overrightarrow{AB}$.

$3\overrightarrow{AB} = \begin{bmatrix} 15 \\ -6 \end{bmatrix}$ and $\frac{1}{2}\overrightarrow{AB} = \begin{bmatrix} 2.5 \\ -1 \end{bmatrix}$.

EXAMPLE 5

Simplify the vector $\begin{bmatrix} -21 \\ 28 \end{bmatrix}$.

We see that the numbers here have a common factor of 7, which we can therefore take out: $\begin{bmatrix} -21 \\ 28 \end{bmatrix} = 7\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

EXERCISE 3

If $\overrightarrow{PQ} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$ and $\overrightarrow{ST} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$ find column vectors for:

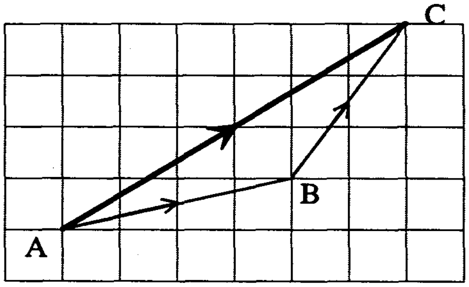
a $5\overrightarrow{PQ}$ **b** $3\overrightarrow{ST}$ **c** $20\overrightarrow{ST}$ **d** $\frac{1}{3}\overrightarrow{PQ}$ **e** $-2\overrightarrow{ST}$ **f** $-\frac{1}{2}\overrightarrow{PQ}$

Simplify: **g** $\begin{bmatrix} 35 \\ 20 \end{bmatrix}$ **h** $\begin{bmatrix} -14 \\ 7 \end{bmatrix}$ **i** $\begin{bmatrix} 52 \\ 44 \end{bmatrix}$ **j** $\begin{bmatrix} -12 \\ -15 \end{bmatrix}$ **k** $\begin{bmatrix} 18 \\ -24 \end{bmatrix}$ **l** $\begin{bmatrix} -100 \\ -120 \end{bmatrix}$

ADDING VECTORS

EXAMPLE 6

Suppose that a point is translated by the vector $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and then translated again by the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.



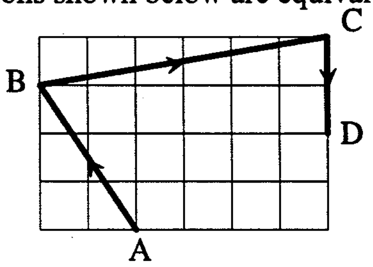
Starting at A the first translation takes you to B and from B the second translation takes you 2 units to the right of B and 3 units up, which brings you to C.

You can see from the diagram that a translation from A to B and then from B to C is the same as a single translation from A directly to C, as shown by the bold line.

We can write this as $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

EXAMPLE 7

Similarly the three translations shown below are equivalent to a single translation.



The translations are from A to B, then from B to C and then from C to D.
The translation from A directly to D has the same result, so

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}.$$

If we write down the column vectors for $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$ and $\overrightarrow{AD}$ we get:

$$\overrightarrow{AB} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \overrightarrow{BC} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}, \overrightarrow{CD} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \overrightarrow{AD} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

If we now look at the top numbers in these four vectors we see that the first three add up to the fourth: -2 + 6 + 0 = 4.
Similarly for the bottom numbers we find: 3 + 1 + -2 = 2.

This shows us how to add vectors:

> Vectors are added by adding the top numbers for the top number of the answer and adding the bottom numbers for the bottom number of the answer.

2 Look at Example 6. Write down the column vectors for $\overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{AC}$.
Add the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ as shown above.
This should be equal to $\overrightarrow{AC}$ which shows that the two routes from A to C are equivalent.
$\overrightarrow{AB} + \overrightarrow{BC}$ is the same as $\overrightarrow{AC}$. Or $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

EXERCISE 4

Add the following vectors:

a $\begin{bmatrix} 7 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

b $\begin{bmatrix} 15 \\ 2 \end{bmatrix} + \begin{bmatrix} 7 \\ 9 \end{bmatrix}$

c $\begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 \\ 13 \end{bmatrix}$

d $\begin{bmatrix} 9 \\ 9 \end{bmatrix} + \begin{bmatrix} -3 \\ -10 \end{bmatrix}$

e $\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -7 \\ 1 \end{bmatrix}$

f $\begin{bmatrix} -4 \\ 5 \end{bmatrix} + \begin{bmatrix} -3 \\ -5 \end{bmatrix}$

g $\begin{bmatrix} 10 \\ -20 \end{bmatrix} + \begin{bmatrix} -15 \\ -15 \end{bmatrix}$

h $\begin{bmatrix} 4 \\ 5 \end{bmatrix} + \begin{bmatrix} -1 \\ -8 \end{bmatrix} + \begin{bmatrix} -4 \\ 5 \end{bmatrix}$

i $\begin{bmatrix} 7 \\ 2 \end{bmatrix} + \begin{bmatrix} 7 \\ 2 \end{bmatrix}$

j $\begin{bmatrix} 6 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

k $3\begin{bmatrix} 7 \\ 11 \end{bmatrix} + \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

l $2\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 4\begin{bmatrix} 4 \\ -3 \end{bmatrix}$

3 Look at the diagram in Example 7 and write down the column vector for DA.

4 Find the column vector for AB + BC + CD + DA.

Your answer should be the zero vector, showing that if vectors form a closed loop (ABCDA) then the sum of the vectors is the zero vector.

The Sutras in use in this chapter are *By Addition and By Subtraction* and *Proportionately.*

40 Simultaneous Equations

Sometimes a problem reduces to a pair of equations, which are "simultaneously" true, rather than a single equation.

In this chapter we look at how to solve such pairs of equations.

EXAMPLE 1

Alice goes into a shop and buys 3 buns and a cup of tea. This costs 65p.
Then Sasha arrives and buys 2 buns and a cup of tea which costs 50p.
What is the cost of a bun and a tea?

Since Alice has a bun more than Sasha the extra 15p she pays must be for that bun. So buns cost 15p. We can then see that if 3 buns and a tea come to 65p and the 3 buns cost 45p, the tea must cost 20p.

Here we have solved a pair of equations:

$$\begin{array}{l} \mathbf{3b + t = 65} \\ \mathbf{2b + t = 50,} \end{array}$$

where b stands for the cost of a bun and t stands for the cost of a tea.

We solved the problem by subtracting the equations:

$$\begin{array}{l} \mathbf{3b + t = 65} \\ \underline{\mathbf{2b + t = 50}} \\ \underline{\mathbf{\ b \qquad = 15}} \end{array}$$

3b - 2b = b,
the t's cancel out
and 65 - 50 = 15.

We then substituted this value for b into the first equation to get the value of t.

● Check that substituting b=15 into the second equation also gives t=20.

We could also add the equations rather than subtract them:

$$\begin{array}{ll} \mathbf{3b + t = 65} & \\ \underline{\mathbf{2b + t = 50}} & + \\ \underline{\mathbf{5b + 2t = 115}} & \end{array}$$

This shows that 5 buns and 2 teas would cost 115p.

Pairs of equations can be added and subtracted, term by term.

EXAMPLE 2

Solve the equations $x + 5y = 17$,
$x + 3y = 13$.

By subtracting the equations the x terms will cancel out:

$$\begin{array}{l} \mathbf{x + 5y = 17} \\ \underline{\mathbf{x + 3y = 13}} \; - \\ \underline{\quad \mathbf{2y = \ 4}} \end{array}$$

Therefore y=2
and substituting this into the second equation we get $x + 3\times2 = 13$, so $x = 7$.
The answer is therefore $\underline{x = 7, y = 2}$.

EXAMPLE 3

Solve $7x + 2y = 37$,
$4x - 2y = 18$.

Here we see **+2y** and **-2y**.
These will be eliminated by **adding** the equations:

$$\begin{array}{l} \mathbf{7x + 2y = 37} \\ \underline{\mathbf{4x - 2y = 18}} + \\ \underline{\mathbf{11x \quad = 55}} \end{array}$$

This gives x=5
and substituting into the first equation above we get $35 + 2y = 37$, so y=1.
Therefore $\underline{x = 5, y = 1}$.

If the coefficients of x or y have the same number and **same sign- subtract** the equations.
If the coefficients of x or y have the same number and **opposite signs- add** the equations.

EXERCISE 1

Solve the following pairs of simultaneous equations, using addition or subtraction:

a $2x + y = 7$, $x + y = 5$

b $3x + y = 10$, $2x + y = 8$

c $x + 3y = 13$, $x + 2y = 10$

d $7x + y = 22$, $2x + y = 7$

e $x + 6y = 26$, $x + 4y = 20$

f $x + 5y = 23$, $x + y = 7$

g $5x + 2y = 14$, $5x + y = 12$

h $9x + 7y = 86$, $x + 7y = 22$

i $3x + y = 19$, $2x - y = 6$

j $5x + 2y = 25$, $4x - 2y = 2$

k $2x + 3y = 17$, $x - 3y = 4$

l $-4x + 5y = 88$, $4x + 3y = 72$

EXAMPLE 4

Solve $5x + 2y = 29$,
$5x - 3y = 19$.

We see that we need to subtract here to eliminate x:

$$\begin{array}{r} \mathbf{5x + 2y = 29} \\ \underline{\mathbf{5x - 3y = 19}} \\ \underline{\mathbf{5y = 10}} \end{array}$$

We need to be careful here as we get

$$2y - - 3y = 5y:$$

we get two negatives that make a plus.

So $y = 2$ and we get $x = 5$ by substitution.

EXERCISE 2

Solve:

a $3x + 4y = 22$ $3x - 2y = 16$	**b** $x + 5y = 32$ $x - y = 2$	**c** $4x + y = 19$ $-3x + y = 5$

Sometimes we can use both addition and subtraction in the same sum.

EXAMPLE 5

Solve $x + y = 6$,
$x - y = 2$.

These equations are actually easy to solve because adding them will eliminate y and subtracting them will eliminate x:

$$\begin{array}{l} \mathbf{x + y = 6} \\ \underline{\mathbf{x - y = 2}} \; + \\ \underline{\mathbf{2x \quad = 8}} \quad \text{so } \underline{x = 4} \end{array} \qquad \begin{array}{l} \mathbf{x + y = 6} \\ \underline{\mathbf{x - y = 2}} \; - \\ \underline{\mathbf{2y = 4}} \quad \text{so } \underline{y = 2}. \end{array}$$

Note in the second sum that the two minuses make a plus: $y - - y = 2y$.

So the answer is $x=4$ and $y=2$.

You can check that this is correct by substituting these values into the two given equations: $4 + 2 = 6$ and $4 - 2 = 2$.

We see here the Vedic Sutras *By Addition and by Subtraction* and *By Alternate Elimination and Retention* both working in the same sum.

EXAMPLE 6

Solve $3x + 2y = 21,$
$3x - 2y = 9.$

The same method works here also:

$$\begin{array}{ll} \mathbf{3x + 2y = 21} & \\ \underline{\mathbf{3x - 2y = 9}} \ + & \\ \underline{\mathbf{6x \quad = 30}} & \text{so } \underline{x = 5} \end{array} \qquad \begin{array}{ll} \mathbf{3x + 2y = 21} & \\ \underline{\mathbf{3x - 2y = 9}} \ - & \\ \underline{\quad \mathbf{4y = 12}} & \text{so } \underline{y = 3} \end{array}$$

EXERCISE 3

Solve:

a $x + y = 10$
$x - y = 4$

b $x + y = 22$
$x - y = 4$

c $x + y = 50$
$x - y = 16$

d $2x + 3y = 16$
$2x - 3y = 4$

e $5x + 2y = 42$
$5x - 2y = 18$

f $3x + 4y = 32$
$3x - 4y = 4$

PROPORTIONATELY

EXAMPLE 7

Solve $3x + 2y = 120,$
$x + y = 50.$

Here adding or subtracting the equations does not eliminate a letter as it did before.

However, by multiplying the second equation by 2 we get $\mathbf{2x + 2y = 100}$.
And now this equation has $\mathbf{2y}$, the same as the first equation above, so we can subtract these and hence eliminate y:

$$\begin{array}{l} \mathbf{3x + 2y = 120} \\ \underline{\mathbf{2x + 2y = 100}} \ - \\ \underline{\mathbf{\ x \qquad = 20}} \end{array}$$

Adding the equations does not give us y however.
But substituting $x=20$ into the second equation above gives $y=30$.
So the answer is $\underline{x = 20,\ y = 30}$.

1 Check that these values also satisfy the first equation above.

2 Solve the pair of equations above again by multiplying the second equation by 3 instead of by 2. This will eliminate x instead of y.

> By multiplying an equation through by some number we can get a term the same as in the other equation. Then subtracting will eliminate that term.

EXAMPLE 8

Solve $5x + 7y = 67,$
$x + 3y = 23.$

Looking at the x terms we see that by multiplying the second equation by 5 we will have two equations with 5x, which we can then subtract:

$$\begin{array}{rl} \mathbf{5x + 15y = 115} & \text{(this is the second equation multiplied by 5)} \\ \underline{\mathbf{5x + \ 7y = \ 67}} & - \\ \underline{\mathbf{8y = \ 48}} & \text{therefore } \underline{y=6}. \end{array}$$

Note that we put the equation with the larger number of y's on top when we subtract as this makes it easier.

Putting y=6 in $x + 3y = 23$ we get $x + 18 = 23$ so that $\underline{x = 5}$.

EXAMPLE 9

Solve $4x - y = 21,$
$5x + 2y = 49.$

Looking at the numbers we see that we have -y and +2y.
This suggests multiplying the first equation by 2 and adding:

$$\begin{array}{rl} \mathbf{8x - 2y = 42} & \\ \underline{\mathbf{5x + 2y = 49}} & + \\ \underline{\mathbf{13x \quad = 91}} & \text{so } \underline{x = 7} \end{array}$$

Putting x=7 into the second equation gives $35 + 2y = 49$ so that $\underline{y = 7}$.

EXERCISE 4

Solve the following:

a $x + 3y = 11$ $2x + 5y = 19$	**b** $x + 2y = 11$ $2x + 3y = 18$	**c** $x + 4y = 24$ $2x + 3y = 23$	**d** $3x + 4y = 17$ $x + y = 5$	**e** $3x + y = 18$ $4x + 2y = 26$
f $5x + 2y = 19$ $2x - y = 4$	**g** $3x + 2y = 25$ $2x - y = 12$	**h** $2x + 3y = 22$ $3x - y = 11$	**i** $2x + 3y = 23$ $x - 2y = 1$	**j** $3x + 4y = 31$ $x - y = 1$

So the basic method we use to solve simultaneous equations is addition and subtraction to eliminate one or both of the letters. Substitution can always be used once one of the letters is known.

We can also use the *Proportionately* formula to help in this elimination process. The following exercise contains a mixture of the various types.

EXERCISE 5

Solve the following simultaneous equations:

a $x + y = 30$
$x - y = 8$

b $2x + y = 23$
$x + y = 16$

c $3x + 2y = 25$
$x - 2y = 3$

d $2x + 3y = 23$
$2x + y = 21$

e $5x + 2y = 37$
$4x + y = 26$

f $3x + 2y = 61$
$3x - 5y = 26$

g $2x + 3y = 54$
$2x - 3y = 30$

h $x + 7y = 46$
$3x - 2y = 23$

i $x + y = 25$
$x - y = 6$

j $5x - 2y = 97$
$4x + 6y = 89$

k $3x + y = 18$
$5x + 2y = 36$

l A pad of paper and a pencil cost 61p. The pad costs 27p more than the pencil. Write down two simultaneous equations using x for the cost of the pad and y for the cost of the pencil. Solve the equations to find the cost of the pad and the cost of the pencil.

m 5 lb of potatoes and 2 lb of carrots cost £1.32. 3 lb of potatoes and a pound of carrots cost 76p. How much do potatoes cost per pound and how much do carrots cost per pound?

SOLUTION BY SUBSTITUTION

Another way of solving simultaneous equations is by substitution.

EXAMPLE 10

Solve $y = 2x + 3$ and $y = 24 - x$.

Since $y = 2x + 3$ we can substitute $2x + 3$ for y in the second equation, this gives:
$2x + 3 = 24 - x$.
And equations like this are easy to solve mentally, giving, in this case, $\underline{x = 7}$.
Then putting $x=7$ into either of the given equations we get $\underline{y = 17}$.

EXERCISE 6

Solve by substitution:

a $y = 4x + 5, y = 2x + 17$ **b** $y = 5x - 1, y = x + 7$ **c** $y = 2x + 5, y = 5x - 19$

d $y = 3x - 1, y = x - 7$ **e** $y = 5x - 2, y = 2x + 5$

TWO SPECIAL TYPES

EXAMPLE 11

Solve $19x - 7y = 119$,
$7x - 19y = 11$.

Notice the patterns in the coefficients of x and y here.
When we see such a pattern as this then two applications of *By Addition and By Subtraction* will solve the equations.

Adding the equations gives: $26x - 26y = 130$, which on division by 26 gives
$\mathbf{x - y = 5}$.
Subtracting the equations gives: $12x + 12y = 108$, which on division by 12 gives
$\mathbf{x + y = 9}$.

We now apply addition and subtraction to $x + y = 9$,
$x - y = 5$ to get $\underline{x = 7, y = 2}$.

EXAMPLE 12

Solve $3x + 2y = 6$,
$9x + 5y = 18$.

This comes under the Vedic Sutra *If One is in Ratio the Other One is Zero.*
We notice that the ratio of the x coefficients is the same as the ratio of the coefficients on the right-hand side: **3:9 = 6:18**.
This tells us that since x is in ratio the other one, y, is zero: **y = 0**.
If y=0 we can easily find x by putting y=0 in the first (or the second) equation:
$3x + 0 = 6$. Therefore x=2.
So $\underline{x = 2, y = 0}$.

So look out for where the ratio of the x or y coefficients is the same as the ratio on the right: the other one is zero.

EXERCISE 7

Solve:

a
$$\begin{aligned} 4x - 3y &= 18 \\ 3x - 4y &= 10 \end{aligned}$$

b
$$\begin{aligned} 7x - 4y &= 48 \\ 4x - 7y &= 18 \end{aligned}$$

c
$$\begin{aligned} 6x - 5y &= 27 \\ 5x - 6y &= 17 \end{aligned}$$

d
$$\begin{aligned} 9x - 4y &= 88 \\ 4x - 9y &= 3 \end{aligned}$$

e
$$\begin{aligned} 3x + 2y &= 6 \\ 2x + 3y &= 9 \end{aligned}$$

f
$$\begin{aligned} 3x + 8y &= 6 \\ 5x + 3y &= 10 \end{aligned}$$

g
$$\begin{aligned} 4x - y &= 20 \\ x + 5y &= 5 \end{aligned}$$

h
$$\begin{aligned} 13x + 17y &= 2 \\ 9x + 51y &= 6 \end{aligned}$$

41 Divisibility and Simple Osculators

We already know how to tell if a number is divisible by 2, 3, 4, 5, 6, 8, 9, 10 and numbers like 6 and 15 which can be expressed as a product of two or more numbers which are relatively prime. Here we see how to test for divisibility by larger numbers, and especially prime numbers.

THE EKADHIKA

You will recall that the Ekadhika is the number "one more" than the one before.
So for example for 19 the Ekadhika is 2 because the one before the 9 is 1 and one more than 1 is 2. For 69 the Ekadhika is 7.

In this chapter the Ekadhika is the number *One More Than the One Before* when the number ends in a 9 or a series of 9's.

So for 13 the Ekadhika is 4 because to obtain a 9 at the end we must multiply 13 by 3 which gives 39 for which the Ekadhika is 4.

Similarly, for 27, to get a 9 at the end we multiply 27 by 7 which gives 189. So the Ekadhika for 27 is 19 (one more than 18 is 19).

EXERCISE 1

Find the Ekadhika for each of the following numbers:

a 29 **b** 89 **c** 109 **d** 23 **e** 43 **f** 7 **g** 17 **h** 21

OSCULATION

There are two types of osculators: the positive osculator and the negative osculator.
The positive osculator is just the Ekadhika. We will deal with the negative osculator later. A simple example will illustrate the osculation procedure.

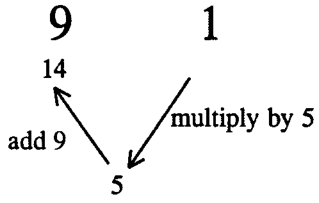
EXAMPLE 1

Find out if **91** is divisible by **7**.

The Ekadhika for 7 is 5, **so we osculate the 91 with 5.**

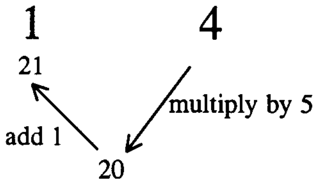
We osculate a number by multiplying its last figure by the osculator and adding the result to the previous figure.

This means we multiply the 1 in 91 by the osculator, 5, and add the result to the 9.



We get **14** as the result.
Since 14 is clearly divisible by 7 we can say that 91 is also divisible by 7.

The result we get from osculation (14 above) can also be osculated: in fact we can continue to osculate as many times as we like.



If we osculate 14 with 5 we get 4X5 + 1 = 21 (also clearly a multiple of 7).
If we osculate this result, 21, we get 1 XS + 2 = 7.
If we osculate 7 (think of 07) we get 7X5 + 0 = 35.

1 Continue this osculation process for four more steps.

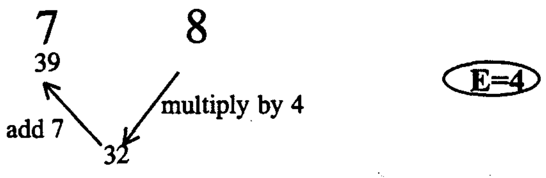
We find that osculating any multiple of 7 with the osculator, 5, always produces a multiple of 7. And osculating any number which is not a multiple of 7 will never produce a multiple of 7.

2 Osculate 16 (which is not a multiple of 7) with the osculator, 5, and continue to osculate until you have produced at least 8 results. None of your results will be a multiple of 7.

EXAMPLE 2

Test **78** for divisibility by **13**.

First we find the Ekadhika for 13, which is 4 (E=4).



Then osculate 78 with this 4: 8×4 + 7 = **39**.
Since 39 is clearly a multiple of 13 we can say that 78 is divisible by 13.

If we did not recognise 39 as a multiple of 13 we could continue to osculate:
For 39: 9×4 + 3 = 39.
Here the 39 gets repeated and this indicates divisibility.
Repetition indicates divisibility.

EXAMPLE 3

Test **86** for divisibility by **13**.

We osculate 86 with **4**: 6×4 + 8 = 32.
This is not a multiple of 13 so 86 is not divisible by 13.

But we can continue osculating if we do not see that 32 is not a multiple of 13.
Osculating the 32 we get 2×4 + 3 = 11.
Clearly 11 is not a multiple of 13, so 86 is not divisible by 13.

EXERCISE 2

Test the following for divisibility by 7:

a 63 **b** 84 **c** 53 **d** 98

Test for divisibility by 13:

e 91 **f** 52 **g** 86 **h** 65

AN EKHADIKA IN NATURAL HABITAT

Test for divisibility by 19:

i 57 **j** 95 **k** 76 **l** 114 (multiply 4 by the osculator and add on 11)

Test for divisibility by 29:

m 87 **n** 116 **o** 57 **p** 58

Test for divisibility by 17:

q 85 **r** 56 **s** 51 **t** 102

TESTING LONGER NUMBERS

If the number we are testing is a long number the osculation procedure is simply extended.

EXAMPLE 4

Is **247** divisible by **19**?

The Ekadhika is **2**.
We multiply the 7 by 2 and add on the next figure, the 4:

$$\begin{array}{ccc} \mathbf{2} & \mathbf{4} & \mathbf{7} \\ & \mathbf{18} & \end{array}$$

7×2 + 4 = 18. We put this under the 4 as shown.

We then multiply this 18 by 2 and add the 2 at the left of 247:

$$\begin{array}{ccc} \mathbf{2} & \mathbf{4} & \mathbf{7} \\ \mathbf{38} & \mathbf{18} & \end{array}$$

18×2 + 2 = 38 which is 2 19's so <u>247 is divisible by 19</u>.

There is a useful short-cut however which prevents the numbers on the bottom line from getting too big.

Suppose the first stage is completed, so that we have:

2 4 7
18

Instead of multiplying 18 by 2 and adding the 2 on, we can suppose that the 1 in the 18 is carried over to join the 2. So we multiply only the 8 by 2 and add the 1 next to it *and* the 2.

So 8 X 2 + 1 + 2 = 19:

2 4 7
19 18

This is much easier and the 19 shows again that 247 is divisible by 19.

EXAMPLE 5

Is **4617** divisible by **19**?

The Ekadhika is still **2**.

We multiply the 7 by 2 and add on the 1:

4 6 1 7
15

Then multiply the 5 by 2 and add on the 1 and the 6:

4 6 1 7
17 15

Then multiply the 7 by 2 and add on the 1 and the 4:

4 6 1 7
19 17 15

We end up with 19 and so 4617 is divisible by 19.

EXERCISE 3

Test the following numbers for divisibility by 19:

a 2774 **b** 589 **c** 323 **d** 4313 **e** 779 **f** 4503

g 14003 **h** 1995 **i** 10203 **j** 30201 **k** 234555

The Ekadhika is 2 for all these. Testing for divisibility for other numbers than 19 means using a different Ekadhika.

EXAMPLE 6

Is **13455** divisible by **23**?

The Ekadhika for 23 is **7**.
We simply osculate as before using this 7:

1	**3**	**4**	**5**	**5**
69	**59**	**8**	**40**	

We have 69 at the end which is clearly 3 23's, so yes: 13455 is divisible by 23.

EXERCISE 4

Test the following numbers for divisibility by the number shown:

a 41963 by 29
b 4802 by 49
c 4173 by 13
d 2254 by 23
e 10404 by 17
f 1003 by 59
g 4171 by 43
h 5432 by 7
i 4321 by 109

OTHER DIVISORS

We can combine our earlier work on divisibility with the osculation method.

EXAMPLE 7

Is **6308** divisible by **38**?

We see that 38 = 2×19 so we have to test for divisibility by 2 **and** 19.
The number is clearly divisible by 2 (because the last figure is even) so we only have to test for 19:

6	**3**	**0**	**8**
19	**16**	**16**	

6308 is also divisible by 19 so it is divisible by 38.

EXAMPLE 8

Is **5572** divisible by **21**?

Since 21 = 3×7 we must test for 3 and 7.
It is not divisible by 3 (digit sum is 1) so we need go no further: 5572 is not divisible by 21.

EXAMPLE 9

Is **1764** divisible by **28**?

We must test for 4 and 7 since 28 = 4×7 (note that 4 and 7 are relatively prime, we do not use 28 = 2×14 as 2 and 14 are not relatively prime).
The last two figures of 1764 indicate that it is divisible by 4 so we test for 7 next.

The osculator is 5: **1 7 6 4**

49 39 26 and we see that the test for 7 is passed.

So 1764 is divisible by 28.

> In testing for divisibility by some number look at the factors of that number first and start with the easiest factors.

EXERCISE 5

Test the following numbers for divisibility by the number shown:

a 3538 by 58
b 1254 by 38
c 21645 by 65
d 1771 by 46
e 767 by 95
f 5985 by 95
g 37932 by 58
h 334455 by 39
i 305448 by 52

THE NEGATIVE OSCULATOR

If we want to test for divisibility by 31 we would have to multiply it by 9 to get a 9 at the end. This gives 279 so that the Ekadhika would be 28. This is too large to osculate easily with so we need an alternative method here and this is where we would use the negative osculator.

The negative osculator for 31 is 3, we just drop the 1.
The negative osculator for 41 is 4. And so on.
To get the negative osculator for 17 we need to get a 1 at the end of the number and this can be done by multiplying 17 by 3: This gives 51 so the negative osculator for 17 is 5.

EXERCISE 6

Obtain the negative osculator for:

a 61 **b** 91 **c** 101 **d** 11 **e** 27 **f** 37

g 7 **h** 13 **i** 23 **j** 19

k Copy and complete the following table showing the positive osculator, P, and the negative osculator, Q, for some numbers.

Number	P	Q	P+Q
1	1	0	
3			
7			
9	1		
11			
13			
17			
19	2	17	19
21			
23			
27	19		

Also fill in the last column by adding P and Q for that row.

You should find a surprising result from this: P + Q = D where D is the divisor, the number being tested.
This is also useful because it means that if we know P we can find Q by taking it from D. And we can find P if Q is known by taking it from D.

EXAMPLE 10

Is **3813** divisible by **31**?

We can use the negative osculator here, which is **3**.
The osculation process is slightly different for the negative osculator.
We begin by putting a bar over every other figure in 3813, starting with the second figure from the right:

$$\bar{3}\quad 8\quad \bar{1}\quad 3$$

We then osculate as normal except that **any carry figure is counted as negative.**

$$\begin{array}{cccc} \bar{3} & 8 & \bar{1} & 3 \\ \mathbf{0} & \mathbf{32} & \mathbf{8} & \end{array}$$

$3\times3+\bar{1}=\mathbf{8}$,
$8\times3+8=\mathbf{32}$,
$2\times3+\bar{3}+\bar{3}=\mathbf{0}$.

This zero indicates that <u>3813 is divisible by 31</u>.

EXAMPLE 11

Test **a** 367164 and **b** 6454 for divisibility by 7?

Multiplying the 7 by 3 gives 21, so we can use a negative osculator of **2**.
a We put a bar on every other figure and osculate as above:

$$\begin{array}{cccccc} \bar{3} & 6 & \bar{7} & 1 & \bar{6} & 4 \\ \mathbf{0} & \mathbf{12} & \mathbf{3} & \mathbf{5} & \mathbf{2} & \end{array}$$

b Similarly:

$$\begin{array}{cccc} \bar{6} & 4 & \bar{5} & 4 \\ \bar{\mathbf{7}} & \mathbf{10} & \mathbf{3} & \end{array}$$

Since $\bar{7}$ is a multiple of 7 we find that <u>both these numbers are divisible by 7</u>.

EXAMPLE 12

Is 11594 divisible by 62?

Since $62 = 31\times2$ we need to test for 31 and 2.
The number is certainly divisible by 2 so we test for 31 by osculating with 3:

$$\begin{array}{ccccc} 1 & \bar{1} & 5 & \bar{9} & 4 \\ \mathbf{0} & \mathbf{10} & \mathbf{14} & \mathbf{3} & \end{array}$$

The number is divisible by 2 and 31 and is therefore divisible by 62.

Similarly, asked if the above number is divisible by 63, since 63 = 9X7 we would check for divisibility by 9 and then (if it passed this test) test for 7 by either using a positive osculator of 5 or a negative osculator of 2.
[We would not use 63 = 3X21 because we would be checking for divisibility by 3 twice and not testing for divisibility by 9. The factors we split 63 into must be relatively prime.]

EXERCISE 7

Test the following for divisibility by the given number:

a 2914 by 31
b 9576 by 21
c 6039 by 61
d 20022 by 71
e 73472 by 41
f 63909 by 81
g 1728 by 91
h 7072 by 17
i 14715 by 27
j 7071 by 61
k 178467 by 31
l 45787 by 7
m 2394 by 42
n 4838 by 82
o 17949 by 93
p 9658 by 11

42 Square Roots

The general method of finding the square root of a number is just the reverse of the squaring process so we begin this chapter by revising squaring.

SQUARING

We square a number by combining the Duplexes contained in the number.

EXAMPLE I

Find 5431^2.

The Duplexes are:
D(5)=**25**, D(54)=**40**, D(543)=**46**, D(5431)=**34**, D(431)=**17**, D(31)=**6**, D(1)=**1**.

Combining these: $2\ 5\underset{\cup}{,}4\ 0 = 290,$
$29\ 0\underset{\cup}{,}46 = 2946,$
$294\ 6\underset{\cup}{,}34 = 29494,$
$2949\ 4\underset{\cup}{,}17 = 294957$
$294957, 6, 1 = \underline{29495761}.$

EXERCISE 1

Square the following numbers (do as many in your head as you can):

a 23 **b** 34 **c** 54 **d** 61 **e** 421 **f** 124 **g** 423

h 818 **i** 4321 **j** 6032 **k** 5234 **l** 4444 **m** 30351 **n** 71017

FIRST STEPS

If we are given a number which we are to find the square root of there are two important facts we can immediately get from the number:

the number of figures in the square root before the decimal point,
the first figure of the square root.

EXAMPLE 2

Suppose we want the square root of 5432.

We mark off pairs of digits from the right 54'32.
Since there are 2 groups of digits formed there will be 2 figures in the square root before the decimal point.

Since the group on the left is 54 this tells us that the first figure will be 7 because **the first square number below 54 is 49** ($=7^2$).

Using these two results together we can say that since the answer starts with 7 and has 2 figures before the decimal point, $\sqrt{5432} \approx \underline{70}$.

You can see that this is correct because $70^2 = 4900$ and $80^2 = 6400$ so the number whose square is 5432 must be between 70 and 80 and must therefore must be seventy something.

EXAMPLE 3

Find an approximate answer for $\sqrt{8,124,569}$.

Split the number into pairs starting at the right: 8'12'45'69.
There are 4 groups so there will be 4 figures in the answer before the point.

The group at the left is 8 and the first square number below 8 is 4, which is 2^2.
So the first figure of the square root is 2.

So the square root begins with 2 and has 4 figures before the point.
Therefore $\sqrt{8,124,569} \approx \underline{2,000}$.

Again we can check this is right by squaring 2,000: $2,000^2 = 4,000,000$,
$3,000^2 = 9,000,000$,
and 8,124,569 lies between 4,000,000 and 9,000,000.

Actually the answer is closer to 3000 than 2000, but we are interested in the first figure rather than the answer to 1 significant figure.

EXAMPLE 4

Find $\sqrt{166177.8}$.

There are 3 groups counting from the decimal point (16'61'77.8) and therefore 3 figures.
The square root of 16 is 4, so this will be the first figure.
Hence $\sqrt{166177.8} \approx \underline{400}$.

EXERCISE 2

Find the number of figures before the decimal point and the first figure of the square root of the following numbers and hence write down an approximate answer:

a 2678 **b** 5277 **c** 527788 **d** 654 **e** 917 **f** 7531 **g** 3700

h 880000 **i** 56789 **j** 234 **k** 2345 **l** 1020304 **m** 77 **n** 11

o 20.34 **p** 580000 **q** 76000 **r** 9000 **s** 7000

SQUARE ROOT OF A PERFECT SQUARE

EXAMPLE 5

Find the square root of 1849.

Marking off two figures from the right, 18'49, we expect two figures before the decimal point and the first figure of the answer is 4.

We set the sum up like this:

$$\begin{array}{r|cccc} & \mathbf{1} & \mathbf{8} & \mathbf{4} & \mathbf{9} \\ 8) & & & 2 & \\ \hline & & \mathbf{4} & & \\ \hline \end{array}$$

Since $4^2 = 16$ and 18 is 2 more than this we have a remainder of 2 which we place as shown.
Note the 24 formed diagonally by this 2 and the 4 above it.
The answer goes on the bottom line.

We also put **twice the first figure**, which is 8, as a divisor at the left as shown.

Next we divide the 24 by the divisor 8.
This gives 3 remainder 0, placed as shown below:

$$\begin{array}{r|cccc} & \mathbf{1} & \mathbf{8} & \mathbf{4} & \mathbf{9} \\ 8) & & & 2 & 0 \\ \hline & & \mathbf{4} & \mathbf{3} & \\ \hline \end{array}$$

We now see 09 and we deduct from this the Duplex of the last answer figure:
D(3) = 9 and 9 - 09 = 0. This means the answer is exactly <u>43</u>.

EXAMPLE 6

Find $\sqrt{1369}$.

Again we expect 2 figures before the point and the first figure will be 3.

$$\begin{array}{r|ccccccc} & \mathbf{1} & & \mathbf{3} & & \mathbf{6} & & \mathbf{9} \\ 6) & & & & 4 & & & \\ \hline & & & \mathbf{3} & & & & \end{array}$$

Since $3^2 = 9$ we have a remainder of 4 placed as shown above. Also we again put twice the first figure, 6, as a divisor on the left.
Note the 46 in the diagonal of figures.

Next we divide the divisor, 6, into 46 and put down 7 remainder 4:

$$\begin{array}{r|ccccccc} & \mathbf{1} & & \mathbf{3} & & \mathbf{6} & & \mathbf{9} \\ 6) & & & & 4 & & 4 & \\ \hline & & & \mathbf{3} & & \mathbf{7} & & \end{array}$$

Finally we see 49 in the second diagonal and we take the Duplex of the last answer figure, D(7) = 49, from this to get 0.

So the answer is exactly 37.

A. We first set up the initial sum including the first figure of the answer, the remainder and twice the first figure placed as a divisor on the left.

B. We then divide the figures shown in the diagonal by the divisor and put down the answer and remainder.

C. Finally check the answer is exact by subtracting the Duplex of the last answer figure from the second diagonal.

EXERCISE 3

Find the square root of the following, using the method shown:

a 3136
b 3969
c 5184
d 3721
e 6889
f 1296
g 2304
h 4624
i 1521
j 3844
k 8649

EXAMPLE 7

Find the square root of 293764.

We first mark off pairs of figures from the right: 29'37'64.
This shows us that we expect 3 figures before the decimal point,
and that the answer begins with a 5.

We set up the sum as before:

```
      2 9 3 7 6 4
  10)     4
        5     .
```

We have a remainder of 4 from the 29 and twice the first figure is 10.
Since we know there are 3 figures before the point we can insert the point as shown.

We get the next figure of the answer by dividing 10 into the 43.
This gives 4 remainder 3 which we write down as shown below:

```
      2  9  3  7  6  4
  10)     4  3
        5  4     .
```

Next, before we divide 10 into the 37 in the second diagonal we subtract the Duplex of the 4 in the answer from it. The Duplex of 4 is 16.
37 - 16 = 21 and 21 ÷ 10 = 2 remainder 1, which we write down as shown below:

```
      2  9  3  7  6  4
  10)     4  3  1
        5  4  2.
```

Next, before dividing 10 into the 16 in the third diagonal we deduct the duplex of the 42 in the answer. D(42)=16, 16-16=0 and 0÷10=0 rem 0:

```
      2  9  3  7  6  4
  10)     4  3  1  0
        5  4  2 . 0
```

Finally we deduct the duplex of the last answer figure, 2, from the 04 in the fourth diagonal.
This leaves 0 and so the answer is exactly <u>542</u>.

The method is a continuation of the shorter sums done before.
We set out the initial sum as before and divide into the first diagonal as before.
The next three steps involve deducting the Duplex of a the 2nd answer figure,
b the 2nd and 3rd answer figures,
c the 3rd answer figure

from the last diagonal before dividing by the divisor.

EXERCISE 4

Find the square root of:

a 186624 **b** 264196 **c** 400689 **d** 318096

e 119025 **f** 524176 **g** 59049 **h** 197136

i 519841 **j** 375769 **k** 53361

GENERAL SQUARE ROOTS

Next we consider the general case where the square root does not terminate but has an infinite number of figures after the decimal point.
This is just an extension of what we have been doing above.

EXAMPLE 8

Find the first 5 figures and an approximate answer for the square root of 38.

There is clearly 1 figure before the point and it is a 6.
There is also a remainder of 2.
Then 20÷12=1 rem 8:

$$\begin{array}{rl} & \mathbf{3\;8.0\;0\;0\;0\;0} \\ 12) & \underline{\quad 2\;\;8\qquad\qquad} \\ & \underline{\mathbf{6.1}\qquad\qquad\quad} \end{array}$$

From now on **we always deduct the Duplex of all the figures** **<u>after the first</u>** (the 6) **from the diagonal figures** and then divide by 12.

So D(1)=1, 80-1=79, 79÷12=**6** rem 7:

$$\begin{array}{rl} & \mathbf{3\;8.0\;0\;0\;0\;0} \\ 12) & \underline{\quad 2\;\;8\;\;7\qquad\quad} \\ & \underline{\mathbf{6.1\;\;6}\qquad\quad} \end{array}$$

Then D(16)=12, 70-12=58, 58÷12=4 rem 10:

$$\begin{array}{rl} & \mathbf{3\;8.0\;0\;0\;0\;0} \\ 12) & \underline{\quad 2\;\;8\;\;7\;10\quad} \\ & \underline{\mathbf{6.1\;\;6\;\;4}\quad} \end{array}$$

Then D(164)=44, 100-44=56, 56÷12=4 rem 8:

$$\begin{array}{rl} & \mathbf{3\;8.0\;0\;0\;0\;0} \\ 12) & \underline{\quad 2\;\;8\;\;7\;10\;8} \\ & \underline{\mathbf{6.1\;\;6\;\;4\;\;4}} \end{array}$$

So $\sqrt{38} \approx \underline{6.1644}$.

EXAMPLE 9

Find the first seven figures of $\sqrt{10330130}$.

The procedure is just the same.
10'33'01'30 shows there are 4 figures before the point and the first one is 3:

1 0 3 3 0 1 3 0.0
6) 1
3 .

There is a remainder of 1 and the divisor is 6.

- Copy the initial sum above carefully into your book.

The full sum to seven figures is shown below:

1 0 3 3 0 1 3 0.0
6) 1 1 3 2 4 5 2
3 2 1 4.0 5 2

The Duplexes to be found at each step are D(2), D(21), D(214), D(2140), D(21405).

- Work through the sum in your book and check that you agree with every step shown.

EXERCISE 5

Find the first 5 figures and an approximate answer for the square root of:

a 27.2727 **b** 38.83 **c** 2929 **d** 11.23

e 3737 **f** 123356 **g** 707172 **h** 5000

i Find the first 9 figures of the square root of 26.123456789

j Find the first 8 figures of the square root of 17

As in the case of division sums it is sometimes necessary
to alter an answer figure or to use bar numbers.

EXAMPLE 10

Find the square root of 19.2.

The initial sum is:

$$\begin{array}{ll} & \mathbf{1\ \ 9.2\ \ 0\ \ 0} \\ 8) & \underline{\quad\ \ 3\qquad\ } \\ & \underline{\quad\ \mathbf{4.}\qquad\ } \end{array}$$

The next step is 32÷8=4 rem 0.

But this would mean subtracting 16 from 0 in the next step:

$$\begin{array}{ll} & \mathbf{1\ \ 9.2\ \ 0\ \ 0} \\ 8) & \underline{\quad\ \ 3\ \ 0\qquad} \\ & \underline{\quad\ \mathbf{4.4}\qquad\ } \end{array}$$

Method 1: Anticipating this happening we can avoid the negative numbers by saying 32÷8=3 rem 8 (see the second diagonal below) rather than 4 rem 0.

$$\begin{array}{ll} & \mathbf{1\ \ 9.2\ \ 0\ \ 0\ \ 0} \\ 8) & \underline{\quad\ \ 3\ \ 8\ \ 7\ \ 14} \\ & \underline{\quad\ \mathbf{4.3\ \ 8\ \ 1}\quad} \end{array}$$

Method 2: Alternatively we can accept the negative numbers:

$$\begin{array}{ll} & \mathbf{1\ \ 9.2\ \ 0\ \ 0\ \ 0\ \ 0} \\ 8) & \underline{\quad\ \ 3\ \ 0\ \ 0\ \ 0\ \ 4\ } \\ & \underline{\quad\ 4.4\ \ \bar{2}\ \ 2\ \ \bar{3}\quad} = \underline{\mathbf{4.3817}} \end{array}$$

After 32÷8=4 rem 0 in the second diagonal we have

D(4)=16, 0-16=$\overline{16}$, $\overline{16}$÷8=$\bar{2}$ rem 0.

Then D(4$\bar{2}$)=$\overline{16}$, 0-$\overline{16}$=16, 16÷8=2 rem 0.

And D(4$\bar{2}$2)=20, 0-20=$\overline{20}$, $\overline{20}$ ÷ 8 = $\bar{3}$ rem 4 (or $\bar{2}$ rem $\bar{4}$).

- Write out the initial sum and work through both these methods to check that you agree with all the steps.

It may take a while to get accustomed to using the bar numbers but they do often simplify the calculation.

Sometimes it is convenient to introduce a bar number right at the beginning of a calculation, when the given number is just below a perfect square.

EXAMPLE 11

Find the first 5 figures of $\sqrt{34}$.

If we begin like this:

$$\begin{array}{r|l} & \mathbf{3\ \ 4\,.\,0\ \ 0\ \ 0} \\ 10) & \underline{\ \ \ \ \ \ 9\ \ 10\ \ \ } \\ & \underline{\ \ \ \ \mathbf{5\,.\,8}\ \ \ \ \ \ } \end{array}$$

the large 8 here leads to large Duplex values and therefore frequent reduction of answer figures.

Alternatively, since 34 is close to 36, a perfect square, we could put 6 for the first figure and a remainder of $\bar{2}$:

$$\begin{array}{r|l} & \mathbf{3\ \ 4\,.\,0\ \ \ 0\ \ 0\ \ \ 0\ \ 0} \\ 12) & \underline{\ \ \ \ \ \ \bar{2}\ \ 4\ \ 0\ \ 0\ \ \bar{5}\ } \\ & \underline{\ \ \ 6\,.\,\bar{2}\ \ 3\ \ 1\ \ 0\ \ \ } \end{array} \quad = \underline{\mathbf{5.8310}}$$

- Check through this sum.

EXERCISE 6

Find the first 4 figures in the square root of the following numbers (do at least the first 4 sums by both methods and remove any bar numbers from your answer).
For the last four use the method of Example 11.

a 27

b 39.6

c 1930

d 11.5

e 575

f 53

g 5

h 2

i 35

j 24

k 3

l 8.321

A puzzle from Mahavira who lived in India about 850BC:
"One third of a herd of elephants and three times the square root of the remaining part were seen on a mountain slope, and in a lake was seen a tusker (male elephant) along with three female elephants. How many elephants were there?"

43 Quadratic Equations

We have already solved some elementary types of quadratic equation in an earlier Chapter.
For example the solution to $3x^2 - 4 = 56$ is $x = \pm\sqrt{20}$.

In this chapter we look at two ways of solving quadratic equations.

EXAMPLE 1

Multiply the binomials $(2x + 3)(x + 2)$.

We get $2x^2 + 4x + 3x + 6 = \underline{2x^2 + 7x + 6}$.

Two binomials produce a quadratic when multiplied together.
The opposite process of converting a quadratic expression like $\mathbf{2x^2 + 7x + 6}$ into $\mathbf{(2x+3)(x+2)}$ is called **factorising the quadratic**.

This is similar to factorising numbers when, for example, we write 21 into 3X7.

First we will practice multiplying some binomials. Then we will see that quadratic expressions can be classified into three types.

EXERCISE 1

Multiply the following mentally to produce quadratic expressions:

a $(x + 5)(x + 6)$
b $(2x + 3)(x + 7)$
c $(5x + 4)(2x + 3)$
d $(x - 5)(x - 4)$
e $(3x - 1)(x - 8)$
f $(7x - 2)(5x - 2)$
g $(x + 3)(x - 6)$
h $(2x - 1)(x + 4)$
i $(3x + 2)(2x - 5)$

If you look carefully at the answers to the last exercise you should see that:

1. from a, b, c- if both of the signs in the binomials are positive then the signs in the quadratic are all positive,

   e.g. $(x + 5)(2x + 3) = 2x^2 + 13x + 15$;

2. from d, e, f- if both of the signs in the binomials are negative then the signs in the quadratic are -, +,

   e.g. $(x - 5)(x - 1) = x^2 - 6x + 5$;

3. from g, h, i- if one of the signs is positive and one is negative then the last sign in the quadratic is negative

   e.g. $(x + 3)(x - 5) = x^2 - 2x - 15$
   or $(x - 3)(x + 5) = x^2 + 2x - 15$.

This means we can predict the signs in the brackets.
For example we know that $x^2 + 9x + 18 = (\quad + \quad)(\quad + \quad)$ the signs will both be plus,
$2x^2 - 7x + 3 = (\quad - \quad)(\quad - \quad)$ the signs will both be minus,
$x^2 + 4x - 21$ or $2x^2 - 5x - 3 = (\quad + \quad)(\quad - \quad)$ one sign is plus and one is minus.

FACTORISING QUADRATIC EXPRESSIONS

Factorise $\mathbf{x^2 + 9x + 18}$.

We need to find the contents of the two brackets in

$$\mathbf{x^2 + 9x + 18 = (\qquad)(\qquad)}.$$

We look at the coefficients in the quadratic: $\mathbf{1x^2 + 9x + 18}$. I.e.. the **1, 9** and **18.**

And **we split the middle coefficient**, 9, into two parts **so that the ratio of the first coefficient to the first part is the same as the ratio of the second part to the last coefficient.**

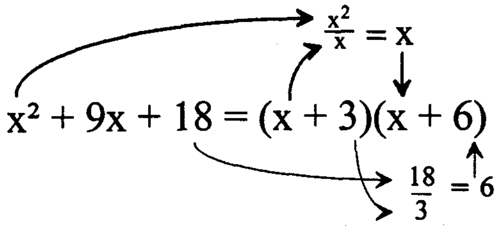
In this case the 9 is split into 3+6: $\mathbf{1x^2 + 9x + 18}$
1 : 3 6 : 18

because this gives the equal ratios 1:3 and 6:18.

This ratio gives us one of the factors we are looking for: it is **(1x + 3)**, or just (x + 3) in which the ratio of coefficients is **1:3**. This is an application of the *Proportionately* Sutra.

So we now have $\mathbf{x^2 + 9x + 18} = (x + 3)(\quad)$.

The other bracket is found using the Sutra *The First by the First and the Last by the Last*:
We divide the first term on the left by the first term in the known bracket and put the result as the first term of the other bracket.
And we divide the last term on the left by the last term in the known bracket to get the last term of the other bracket:



The middle coefficient which was split into 3+6 could also have been split into 6+3.

$$\mathbf{1x^2 + 9x + 18}$$
$$\mathbf{1 : 6 \quad 3 : 18}$$

This also gives the same ratio, 1:6, and the second factor (x + 6).

EXAMPLE 3

Factorise $2x^2 + 13x + 15$.

The coefficients are 2, 13 and 15 and we split the 13 in the same way as we did above to obtain two equal ratios:

$$\mathbf{2x^2 + 13x + 15}$$
$$\mathbf{2 : 10 \quad 3 : 15}$$

We see here that if we split 13 into 10+3 we get ratios of **1:5** and so this gives us a factor:

$$\mathbf{2x^2 + 13x + 15 = (x + 5)(\quad)}.$$

Then $2x^2 \div x = \mathbf{2x}$ and $15 \div 5 = \mathbf{3}$ so $\underline{\mathbf{2x^2 + 13x + 15 = (x + 5)(2x + 3)}}$.

There is a useful checking device for this very similar to the digit sum checking method. We apply the formula *The Product of the Sum of the in the Factors is equal to the Sum of the Coefficients in the Product.*

For example, we have above: $(1x + 5)(2x + 3) = 2x^2 + 13x + 15$.
The coefficients in the factors are added: 1+5=**6**, 2+3=**5** and the product of these is **30**.
But the sum of the coefficients in the product (the right-hand side) is also 30 as 2+13+15=**30**.
This verifies the answer.

EXERCISE 2

Use this method of splitting the middle coefficient to factorise the following quadratics:

a $2x^2 + 9x + 4$
b $2x^2 + 9x + 9$
c $2x^2 + 13x + 6$
d $x^2 + 11x + 28$
e $x^2 + 8x + 15$
f $x^2 + 8x + 12$
g $2x^2 + 9x + 10$
h $3x^2 + 16x + 5$
i $3x^2 + 7x + 2$
j $x^2 + 6x + 5$
k $3x^2 + 14x + 8$
l $4x^2 + 8x + 3$

EXAMPLE 4

Factorise $2x^2 - 7x + 3$.

The signs in the quadratic here are - , + and so we know from Exercise 1 that this means the signs in the binomials are both negative.
We split the -7 into -1 and -6 to produce equal ratios:

$$\begin{array}{l} \mathbf{2x^2 \ - \ 7x \ + \ 3} \\ \mathbf{2 \ : \ -1 \ -6 \ : \ 3} \end{array}$$

You can see the ratios are equal by dividing both sides of -6:3 by -3.
This gives 2:-1.
This ratio gives one factor as $(2x - 1)$ and using *the First by the First and the Last by the Last* the other factor is $(x - 3)$.
So $2x^2 - 7x + 3 = \underline{(2x - 1)(x - 3)}$.

EXERCISE 3

Factorise the following:

a $x^2 - 8x + 15$
b $x^2 - 8x + 12$
c $x^2 - 10x + 9$
d $2x^2 - 11x + 5$
e $3x^2 - 8x + 4$
f $2x^2 - 11x + 12$

EXAMPLE 5

Factorise $2x^2 - 5x - 3$.

This is an example of the third type: the last sign is negative and so the signs in the binomials are + and - or - and +.

In this example the middle coefficient, -5, is split into 1, -6:

$$\mathbf{2x^2 \; - \; 5x \; - \; 3}$$
$$\mathbf{2 \; : \; 1 \; -6 \; : \; -3}$$

The ratio 2:1 then gives the factor $(2x + 1)$ and the other factor is therefore $(x - 3)$. So $2x^2 - 5x - 3 = \underline{(2x + 1)(x - 3)}$.

EXAMPLE 6

Factorise $3x^2 + 8x - 3$.

This is also the third type:

$$\mathbf{3x^2 + \; 8x \; - \; 3}$$
$$\mathbf{3 \; : \; 9 \; -1 \; : -3}$$

The 8 is split into 9, -1 and the ratio is 1:3.

Therefore $3x^2 + 8x - 3 = \underline{(x + 3)(3x - 1)}$.

EXERCISE 4

Factorise the following:

a $2x^2 + 9x - 5$ **b** $2x^2 - 5x - 12$ **c** $x^2 + x - 12$ **d** $x^2 - x - 12$ **e** $2x^2 + 7x - 15$

f $2x^2 - 3x - 20$ **g** $3x^2 - 2x - 8$ **h** $x^2 + 6x - 27$ **i** $6x^2 + 7x - 3$

The next exercise contains a mixture of all the three types.

EXERCISE 5

Factorise the following:

a $x^2 - 4x + 3$ **b** $x^2 - 4x + 4$ **c** $2x^2 + x - 1$ **d** $x^2 + 5x - 6$ **e** $2x^2 + 11x + 12$

f $x^2 + 10x + 24$ **g** $x^2 + x - 30$ **h** $6x^2 + 7x + 2$ **i** $3x^2 - 10x + 8$

SOLVING QUADRATIC EQUATIONS BY FACTORISATION

EXAMPLE 7

Solve $3x^2 + 8x - 3 = 0$

We first of all factorise the quadratic and get: $(x + 3)(3x - 1)$.

So we now have $(x + 3)(3x - 1) = 0$

Since we have now got two binomials which give 0 when multiplied together it follows that one or both of these binomials must equal 0.

So we get: either $(x + 3) = 0$ or $(3x - 1) = 0$.

And we now have two simple equations to solve, giving $\underline{x = -3 \text{ or } x = \frac{1}{3}}$.

> We can solve quadratic equations by factorising the quadratic, forming two small equations and solving these small equations.

EXAMPLE 8

Solve **a** $x^2 = 9x - 20$ **b** $8x^2 - 2x = 0$ **c** $2x^2 + 5 = 23$

a We take the terms on the right over to the left to get: $x^2 - 9x + 20 = 0$.
Then factorise to get $(x - 4)(x - 5) = 0$.
So $(x - 4) = 0$ or $(x - 5) = 0$ and $\underline{x = 4 \text{ or } x = 5}$.

b There is no need to use the method of splitting the middle coefficient here because this is easy to factorise. we spot that 2x is a common factor and write: $8x^2 - 2x = 2x(4x - 1)$.

So $2x(4x - 1) = 0$ and so either $2x = 0$ or $(4x - 1) = 0$.
Therefore $\underline{x = 0 \text{ or } x = \frac{1}{4}}$.

Equations like $8x^2 - 2x = 0$, **in which there is no third coefficient,** are best solved in this way.

c This need not be solved by factorising but we include it here for completeness. We solved equations like this in an earlier Chapter using *Transpose and Apply.* We take the 5 from the 23, divide by 2 and take the square root. This gives $\underline{x = \pm 3}$.

Equations like $2x^2 + 5 = 23$, **in which there is no x term**, are best solved like this.

EXERCISE 6

Solve the following quadratic equations:

a $x^2 + 8x + 15 = 0$

b $2x^2 + 15x + 18 = 0$

c $3x^2 + 13x + 4 = 0$

d $3x^2 + 11x + 6 = 0$

e $3x^2 - 10x + 8 = 0$

f $x^2 = x + 30$

g $2x^2 - 7x - 15 = 0$

h $3x^2 + 8x = 3$

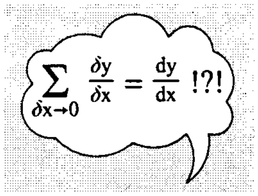
i $4x^2 - 5x - 6 = 0$

j $2x^2 + 3x = 0$

k $3x^2 - 7 = 41$

l $x^2 - x = 0$

DIFFERENTIAL CALCULUS



There is another way of solving quadratic equations by using the Vedic formula *Differential Calculus.*
This involves taking an algebraic expression and obtaining another one from it by a method called differentiation.

EXAMPLE 9

Differentiate $7x^3$.

We differentiate by multiplying the power by the coefficient and putting the answer as the new coefficient. And the new power is the old power minus one.

So in this case we multiply 3 by 7 to get 21 as the coefficient of the answer. ☞

And we reduce the answer, 3, by 1 to 2 to get the power of the answer.

This gives $\underline{D = 21x^2}$ where D means the differentiated expression or differential.

EXAMPLE 10

Differentiate $5x^4 - 2x^3 + x^2$.

We apply the above process to each of the three terms:

$\underline{D = 20x^3 - 6x^2 + 2x}$note that in the 2nd term we multiply the power, 3, by -2 to get -6, and in the 3rd term the coefficient of x^2 is 1.

EXAMPLE 11

Differentiate $5x^2 + 4x - 3$.

Remembering that $x^0 = 1$ we can write this as $5x^2 + 4x^1 - 3x^0$.

So $D = 10x^1 + 4x^0 - 0x^{-1} = \underline{10x + 4}$ because $x^0=1$ and 0 multiplied by anything equals 0.

It follows from this that any number (like -3 above) differentiates to 0.
And any number of x's differentiates to that number, i.e.. the 4x differentiates to 4.

EXAMPLE 12

Differentiate $-17x + 23$.

The 23 differentiates to 0 and the $-17x$ differentiates to -17, so $\underline{D = -17}$.

To differentiate we multiply the power by the coefficient and reduce the power by 1. E.g.. for $7x^2$, $D=14x$.

Any multiple of x differentiates to that multiple
and any number not multiplied by x differentiates to 0. E.g.. for $8x+5$, $D=8$.

EXERCISE 7

Differentiate the following expressions:

a $8x^4$ **b** $3x^{23}$ **c** $4x^2 + 9x + 7$ **d** $5x^3 + 6x^2 - 11x + 4$ **e** $x^3 - 13x$

f $6x^4 + 1$ **g** $12 + 5x^3$ **h** $3 - 5x - 6x^2$ **i** $-8x$ **j** 13 **k** $1 - 5x$

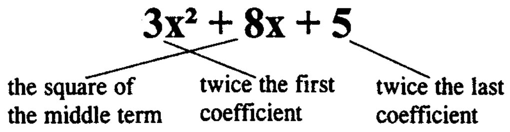
Next we must define the discriminant of a quadratic expression.

The discriminant of a quadratic expression is
the square of the coefficient of the middle term minus the product of twice the coefficient of first (x^2) term and twice the last term.

EXAMPLE 13

Find the discriminant of $3x^2 + 8x + 5$.

According to the above definition the discriminant = $8^2 - (2\times3)\times(2\times5) = \underline{4}$.



$$8^2 \quad - \quad 2 \times 3 \quad \times \quad 2 \times 5$$

EXAMPLE 14

Find the discriminant for $3x^2 - 4x - 5$.

The discriminant = $(-4)^2 - (2\times3)(2\times-5) = 16 - -60 = \underline{76}$.

You will need to be careful with the signs here.

EXERCISE 8

Find the discriminant for:

a $3x^2 + 7x + 2$ **b** $3x^2 - 7x + 2$ **c** $2x^2 + 3x - 5$

d $x^2 - 4x - 6$ **e** $2x^2 - 4x - 8$ **f** $4x^2 - 4x + 1$ **g** $x^2 + 2x + 2$

There is a simple relationship between the differential and the discriminant in a quadratic equation:

> The differential is equal to the square root of the discriminant.

We will see a proof of this important result in a later chapter.

EXAMPLE 15

Solve $x^2 - 8x + 15 = 0$.

The differential is $\mathbf{2x - 8}$ and the discriminant is $\mathbf{4}$.

Since the differential is the square root of the discriminant: $\mathbf{2x - 8} = \pm\sqrt{4}$. So $\mathbf{2x - 8} = \pm 2$.

The ± gives us two equations: $\mathbf{2x - 8 = +2}$ and $\mathbf{2x - 8 = -2}$.

These are easily solved to give $\underline{x = 5 \text{ or } x = 3}$.

EXAMPLE 16

Solve $3x^2 + 8x - 3 = 0$.

The differential is the square root of the discriminant gives $\mathbf{6x + 8 = \pm \sqrt{100}}$.
So $\mathbf{6x + 8 = \pm 10}$.
$\mathbf{6x + 8 = 10}$ gives $x = \frac{1}{3}$. And $\mathbf{6x + 8 = -10}$ gives $\underline{x = -3}$.

EXAMPLE 17

Solve $x2 - 3x - 1 = 0$ a giving the exact answer,
b giving the answer to 2 decimal places.

Examples 15 and 16 above could also have been solved by factorising the quadratic: the first method shown in this chapter.
However not all quadratics will factorise and this is where this second method is most useful.

a The differential is the square root of the discriminant gives $2x - 3 = \pm \sqrt{13}$.
Adding 3 on the right and then halving gives an exact answer of $\underline{x = \frac{1}{2}(\pm \sqrt{13} + 3)}$.

b We obtain $2x - 3 = \pm \sqrt{13}$ as before.
Then we find the square root of 13 to 3 decimal places:

$$\begin{array}{rl} & \mathbf{1\ 3\,.\,0\ 0\ 0\ 0} \\ 6) & \underline{\quad\ \ 4\ \ 4\ \ 4\ \ 4\ } \\ & \underline{\quad \mathbf{3\,.\,6\ 0\ 6}\ \ } \end{array}$$

We now have $2x - 3 = \pm 3.606$ and we can solve these two equations to get:

If $2x-3=3.606$ then $x = 3.303 = \underline{3.30 \text{ to 2 d.p.}}$
and if $2x-3=-3.606$ then $x = -0.303 = \underline{-0.30 \text{ to 2 d.p.}}$

EXERCISE 9

Find exact answers to the following quadratic equations using the differentiation method:

a $x^2 + 6x + 5 = 0$
b $x^2 - 3x - 12 = 0$
c $x^2 - 8x + 15 = 0$
d $2x^2 - 5x - 3 = 0$
e $2x^2 - 5x + 3 = 0$
f $x^2 - 10x + 9 = 0$
g $x^2 - x - 12 = 0$
h $2x^2 - 7x + 3 = 0$
i $2x^2 + x - 6 = 0$
j $x^2 - 2x - 4 = 0$
k $x^2 + 5x - 2 = 0$
l $2x^2 + 8x + 1 = 0$

Find answers to 2 decimal places:

m $x^2 + x - 4 = 0$ **n** $2x^2 + 3x - 4 = 0$ **o** $x^2 + 5x - 7 = 0$ **p** $2x^2 + 7x + 1 = 0$

q $3x^2 + x - 6 = 0$ **r** $x^2 - 8x + 3 = 0$ **s** $x^2 + 10x + 3 = 0$

44 Pythagoras' Theorem

This must be the most famous theorem (law) in mathematics, though it was known long before in India, China and Egypt.

● Take a sheet of graph paper and draw axes along the bottom (shorter side) and left sides. Number up to 18 along the bottom and up to 26 on the left side (every cm is 1 unit on both axes).

● Draw, with a colour, the triangle with vertices (2,2), (4,2), (4,4).
Draw the square (4,2), (6,2), (6,4), (4,4) and the square (4,2), (4,0), (2,0), (2,2). Draw also the square (2,2), (0,4), (2,6), (4,4).

1 Find the area of each of these squares (you can get the area of the largest square by drawing the diagonals in and adding up the areas of each of the 4 small triangles).
Write the area of each square inside the square.

The longest side of a right-angled triangle (opposite the right angle) is called the hypotenuse.

● Draw the triangle (14,4), (10,4), (10,7) in colour.
Carefully draw a square on each side of the triangle (the square on the hypotenuse has vertices at (13,11) and (17,8)).

2 Write the area of each of the smaller squares inside the square.

One way of getting the area of the square on the hypotenuse is to split it up as follows:

Draw a line from (14,4) up to (14,8),
a line from (17,8) across to (13,8),
a line from (13,11) down to (13,7), and
a line from (10,7) across to (14,7).

This splits the large square into four triangles and a square, so you can find its area by adding up the five areas.

3 Write the area of the square on the hypotenuse inside it.

4 In each of the diagrams you have drawn there is a connection between the areas of the three squares. Write down what you think this connection is.

EXERCISE 1

On the same sheet of graph paper draw two more triangles (**a** and **b** below).
Draw the squares on the three sides and find their areas.
Two of the vertices of the square on the hypotenuse are also given in square brackets in case you have difficulty in drawing the large square. You can find the area of the square on the hypotenuse in each case by splitting it into four triangles with a square in the middle, as you did in the previous diagram.

a (3,12), (7,12), (7,14) [(1,16), (5,18)] **b** (9,18), (11,18), (11,24) [(3,20, (5,26)]

5 Do your answers to Exercise 1 confirm the connection you found in 4 above between the areas of the three squares on the sides of each triangle?

The result you should have found is that the areas of the squares on the two shorter sides add up to the area of the square on the hypotenuse.

This is known as the theorem of Pythagoras and it is true for all right-angled triangles. You will shortly see a proof of this theorem.

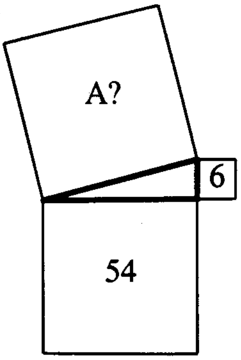
In any right-angled triangle the sum of the areas of the squares on the two shorter sides is equal to the area of the square on the hypotenuse.

Or, using the appropriate Vedic Sutra: *The Sum of the Squares* (on the two shorter sides) *is equal to the Square of the Sum* (the hypotenuse).

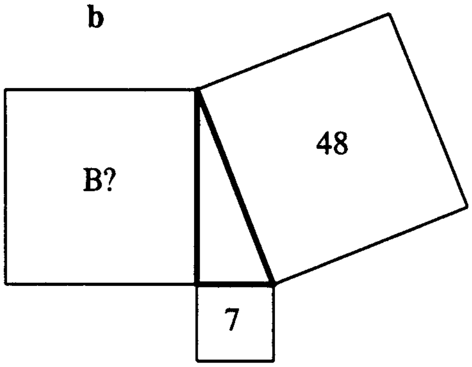
EXERCISE 2

In **a** and **b** below you are given two areas. Find the area A and the area B:

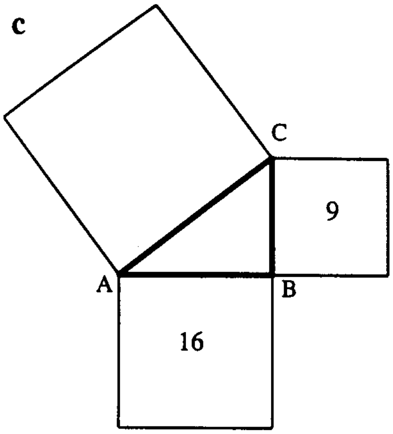
a



b

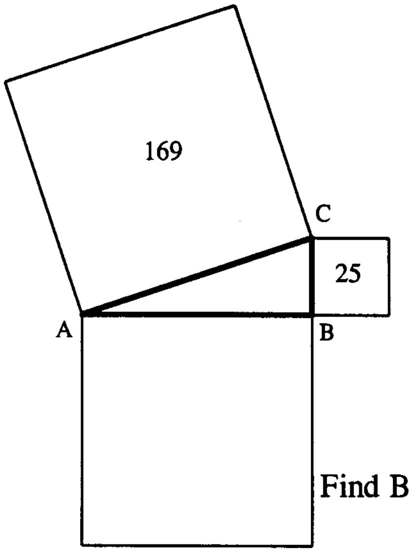


c



Find AB, BC and AC.

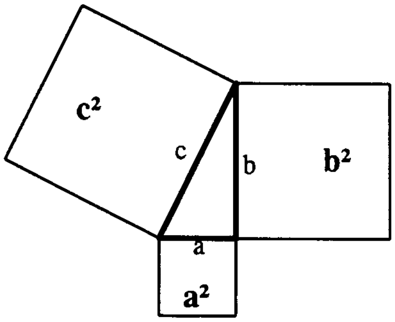
d



Find BC, AC and AB

AN ALGEBRAIC FORMULA

Pythagoras' Theorem has been defined geometrically above- in terms of areas. Another way of describing this theorem is by means of an algebraic formula.



Suppose the right-angled triangle has sides a, b and c as shown above.
Then the areas are a^2, b^2 and c^2 and so $\mathbf{a^2 + b^2 = c^2}$.

Pythagoras' Theorem can also be expressed by the formula:

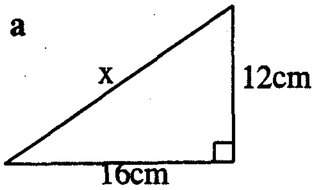
$$a^2 + b^2 = c^2$$

where a, b and c are the sides of a right-angled triangle, c being the hypotenuse.

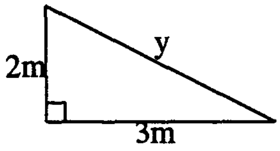
EXAMPLE 1

Find x and y in the triangles below.

a



b



a According to the theorem $x^2 = 12^2 + 16^2$.
Therefore $x^2 = 144 + 256$.
So $x^2 = 400$ and $\underline{x = 20\text{cm}}$. #

b Similarly $y^2 = 3^2 + 2^2$
$y^2 = 13$
$\underline{y = \sqrt{13}}$ m. We can leave the answer like this unless asked to express it in decimals.

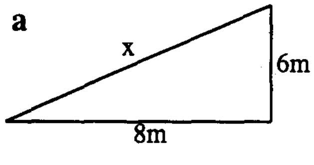
Although $x^2 = 400$ has two solutions (±20), x represents a length and cannot be negative, so the only solution is x=20.

We see then that Pythagoras' Theorem brings together in a simple and elegant way the three main branches of mathematics: Arithmetic, Algebra and Geometry.

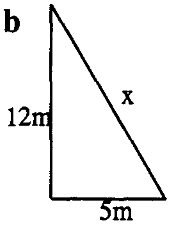
EXERCISE 3

Find x in the right-angled triangles below:

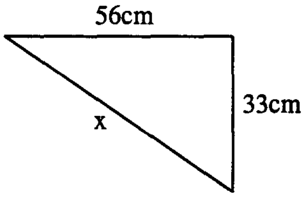
a



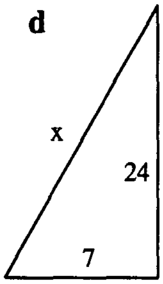
b



c



d



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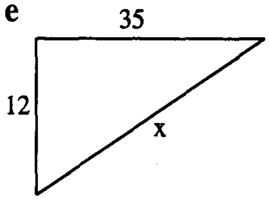
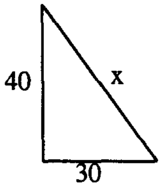


f



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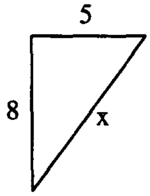

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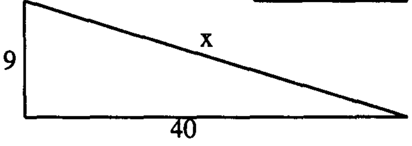
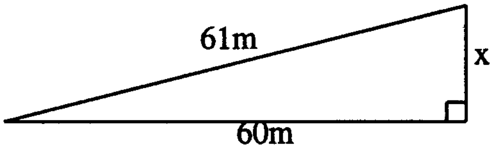


EXAMPLE 2

Find x in the triangle shown.



According to Pythagoras' Theorem $x^2 + 60^2 = 61^2$.
So $x^2 = 61^2 - 60^2$.
Therefore $x^2 = 3721 - 3600 = 121$ and $\underline{x = 11m}$.

This example shows that when one of the shorter sides of the triangle is to be calculated we subtract the squares of the other two sides.

> When Pythagoras' Theorem is used to find the hypotenuse of a right-angled triangle we **add the squares of the other sides and take the square root.**
>
> And when we are finding either of the short sides we **subtract the squares of the given sides and take the square root.**

You may recall from Chapter 2 that there is an easy way of finding the difference of two squares, using *By Addition and By Subtraction*. So if we want $17^2 - 13^2$ we add and subtract 17 and 13:

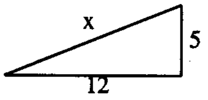
$$17^2 - 13^2 = (17 + 13)(17 - 13) = 30 \times 4 = 120.$$

So in the example above: $61^2 - 60^2 = 121 \times 1 = 121$.
You can use this method in the exercise below if you wish.

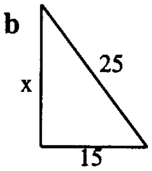
EXERCISE 4

Find x in the following right-angled triangles:

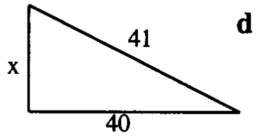
a



b



c



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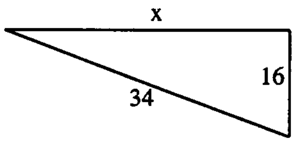

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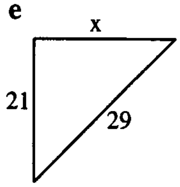

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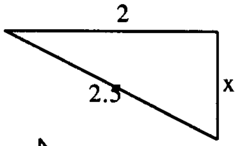

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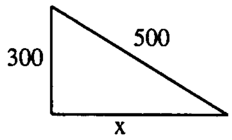

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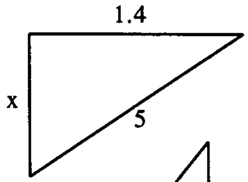

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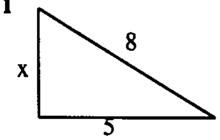

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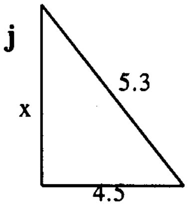

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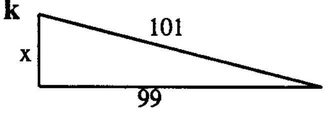

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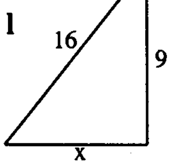
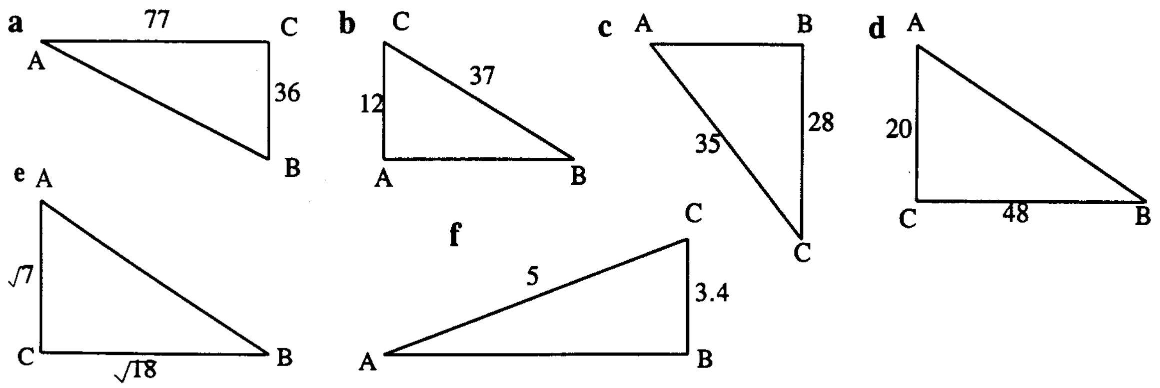


EXERCISE 5

Find AB in each of the following right-angled triangles:

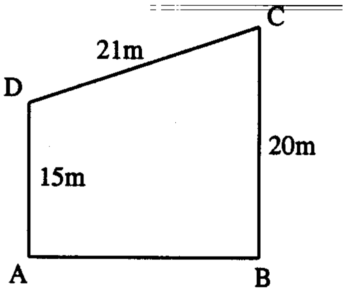


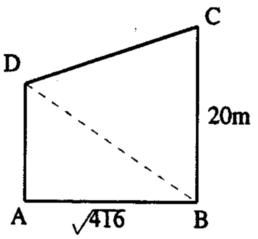
PROBLEMS

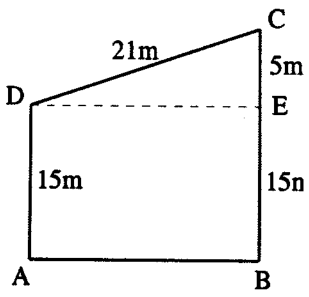
Sometimes a problem leads to the use of Pythagoras' Theorem.

EXAMPLE 3

The diagram shows the view of a wall ABCD.
Find AB and BD.







If we draw a horizontal line across from D
we form a right-angled triangle
in which we know two of the sides.

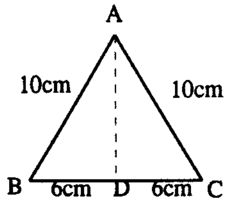
Since the two vertical sides are 15m and 20m we can find CE to be 5m. AB must be the same length as DE so we find DE from the triangle: $DE^2 = 21^2 - 5^2 = 416$

So $AB = \sqrt{416}$ m.

We can use our answer for AB to get BD by using ΔABD:
$BD^2 = (\sqrt{416})^2 + 15^2 = 416 + 215 = 641$.
So $BD = \sqrt{641}$ m.

EXAMPLE 4

In the isosceles triangle ABC, AB=AC=10cm and BC=12cm. Find AD where D is the mid point of BC.
Isosceles triangles can always be divided into two congruent (identical) triangles:

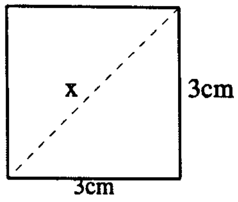


This means we can use Pythagoras' Theorem to find AD.

$AD^2 = 10^2 - 6^2 = 64$.
AD = 8cm.

EXAMPLE 5

Find the length of the diagonal of a square of side 3cm to 2 d.p.



We can draw a diagonal to get a right-angle triangle.

Then $x^2 = 3^2 + 3^2 = 18$.
So $x = \sqrt{18}$.

```
  1 8.0 0 0 0
8)   2 4 4 8
   4.2 4 2
```

The square root calculation shows that the diagonal is 4.24 to 2 d.p.

EXERCISE 6

Solve the following problems:

a Find the length of the diagonal of a rectangle with sides 2½cm and 6cm.

b A walker travels 12km due east from a town A then 9km due south to a town B. How much longer was the journey than if a direct walk from A to B was possible?

c Find the distance of the point with coordinates (7,24) from the Origin.

d In the triangle PQR, PQ=PR=7m and PX=6m where X is the midpoint of QR. Find QR to 2 d.p.

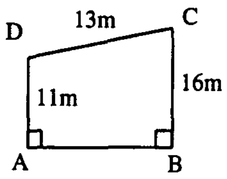
e In the triangle ABC, M is the midpoint of BC and AB=AC. If BC=AM=6 find AC to 2 d.p..

f A watch has hands of length 1cm and 2cm. Find the distance (to 2 d.p.) between the tips of the hands at 3 o'clock.

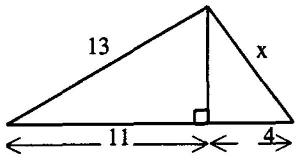
g A ladder 6.5m long is leaning against a vertical wall with its foot 2.5m from the base of the wall. How far up the wall will the ladder reach?

h A triangle has sides 29m, 29m and 40m. Find its area.

i Find AB and AC in this trapezium.



j Find x:



k Find to 2 d.p. the length of the diagonal of a square of side 5cm.

l A square of side x has diagonals of length 8cm. Form an equation and hence find the exact value of x.

THE THEOREM IN REVERSE

Pythagoras' Theorem says that if a triangle is right-angled then the sum of the squares on the two shorter sides equal the square on the hypotenuse.
The reverse is also true:

If the sum of the squares on the two shortest sides of a triangle is equal to the square on the third side then the triangle is right-angled.

EXAMPLE 6

Show that a triangle with sides 12, 9 and 15 is right-angled.

$12^2 + 9^2 = 144 + 81 = 225$.
And $15^2 = 225$.
Since we get 225 in each case the triangle is right-angled.

EXERCISE 7

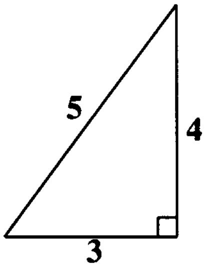
For each of the triangles given below find out which triangles are right-angled and which are not.

a 48, 55, 73 **b** 7, 11, 13 **c** 7, 23, 24 **d** 40, 399, 401

45 Triples

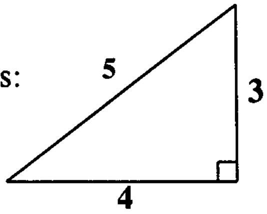
A triple is a set of three numbers which can be the lengths of the sides of a right-angled triangle.

One example of a triple is **3, 4, 5** because if a triangle had sides of 3, 4 and 5 units it would be right-angled ($3^2 + 4^2 = 5^2$):



And let us say that the first number in the triple is always the base of the triangle, the second number is always the height and the third is always the hypotenuse.

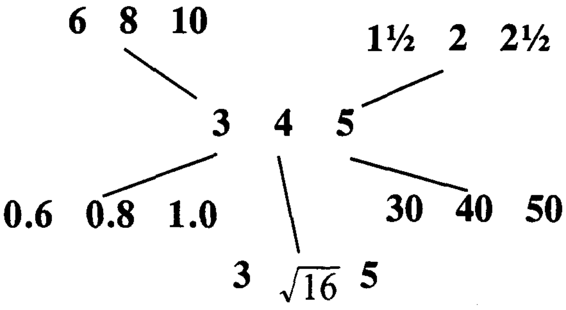
So **4, 3, 5** represents:



Similarly 2, 1½, 2½; $\sqrt{5}, \sqrt{6}, \sqrt{11}$; 2, $\sqrt{5}$, 3 are also triples.

- Confirm that these are triples.

EQUAL TRIPLES



The six triples shown above all have the same shape: 6, 8, 10 for example will have the same shape as 3, 4, 5 because the numbers have all been doubled which enlarges the triangle without changing the shape.

1 What must **a** 1½, 2, 2½,

b 30, 40, 50 ,

c 0.6, 0.8, 1.0 be <u>multiplied</u> by to give 3, 4, 5?

2 Why is $3, \sqrt{16}$, 5 the same as 3, 4, 5?

The six triples above are equal triples because they all have the same shape.

<u>EXAMPLE 1</u>

Similarly 36, 15, 39 = 12, 5, 13 = 6, 2½, 6½.

In the set of six equal triples above 3, 4, 5 is the simplest because it has no fractions and the numbers 3, 4, 5 are relatively prime so the triple cannot be cancelled down any further.
In Example 1 12, 5,13 is the simplest triple.

<u>EXERCISE 1</u>

Find the simplest triple equal to the following triples (you do not need to check that they are triples):

a 24, 10, 26
b 12, 9, 15
c 60, 32, 68
d 42, 40, 58
e 12, 3½, 12½
f 7½, 4, 8½
g 60, 80, 100
h 8.4, 1.3, 8.5
i 8, 3.9, 8.9
j 7, 2.4, 7.4
k 32, 24, 40

TYPES OF NUMBERS

Numbers can be classified into various types:

Natural numbers which are positive whole numbers: 1, 2, 3, 4, . . .
Integers which are positive and negative whole numbers and zero:. . . -3, -2, -1, 0, 1, 2, 3, 4, . . .
Rational Numbers which are integers and fractions: e.g. 1½, -5, -¾, 17.

Note that natural numbers are included in the integers and integers are included in the rational numbers.
There are also numbers which do not come into any of these categories, called irrational numbers. Examples of these are $\sqrt{2}$ and π (but $\sqrt{9}=3$ and so the square root of a perfect square is always rational).

PERFECT TRIPLES

If each of the elements (parts) of a triple are rational numbers then the triple is a perfect triple.

However, 3, $\sqrt{7}$, 4 is a triple but not a perfect triple as $\sqrt{7}$ is not a rational number.

And 4, 6, 7 is not a perfect triple as it is not even a triple: $4^2 + 6^2$ does not equal 7^2.

EXERCISE 2

Copy and complete the table below which classifies the following into perfect triples and triples which are not perfect (two of them are not triples)?

12, 3½, 12½

3, $\sqrt{16}$, 6

72, 65, 97

5, 7, $\sqrt{74}$

12, 5, 13

12, 14, 20

$\sqrt{9}$, 4, 5

$\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$

$\sqrt{14}$, 4, $\sqrt{30}$

PERFECT TRIPLES	NON-PERFECT TRIPLES
E.g. 4, 3, 5	E.g. 2, 1, $\sqrt{5}$

You should have added 4 perfect triples to the table.

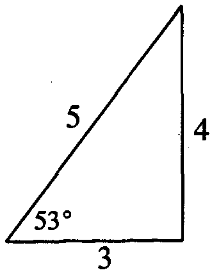
- On a sheet of plain paper draw those 4 perfect triples: choose a suitable scale for each one and draw the base first and then use compasses to find the third vertex. Check that each triangle is right-angled.

3 If you drew out the triples in the second column of your table in the same way would you expect them to be right-angled?

ANGLE IN A TRIPLE

The angle in a triple is the angle between the base and the hypotenuse. So for the 3, 4, 5 triple we can write:

53°) 3, 4, 5 for:

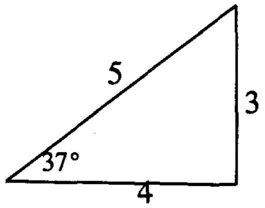


We write the angle first, then a bracket, then the sides of the triple.

❋ Measure the angle in the 3, 4, 5 triple you have drawn (or the one above) to confirm that
angle is about 53°.

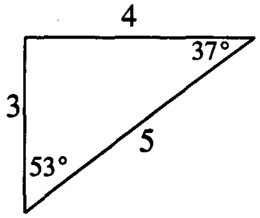
Similarly for the 4, 3, 5 triple, we can write:

37°) 4, 3, 5 for



❋ Measure the angle in the 4, 3, 5 triple to confirm that it is about 37°.

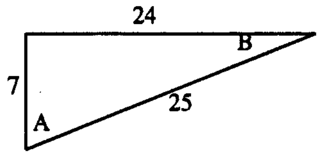
Note that both these triples are contained in one triangle:



If you imagine that 3 is the base you get the triple 53°) 3, 4, 5
and if you take 4 as the base you get 37°) 4, 3, 5.
Every triangle contains two triples.

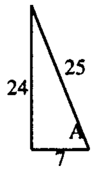
EXAMPLE 2

Write down the triple which has angle A:

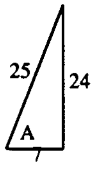


The angle A is between the sides 7 and 25, so 7 must be the base.
We therefore put the 7 first: A)7, 24, 25.

You can also imagine that the triangle is turned so that 7 is the base:



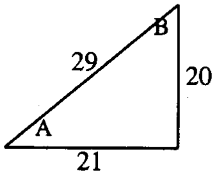
OR

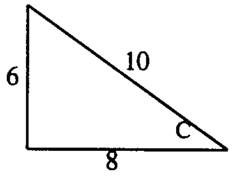


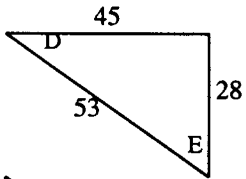
Similarly if the angle B above was to be described in triple terms it would be: B)24, 7, 25.

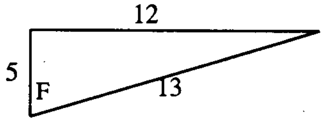
EXERCISE 3

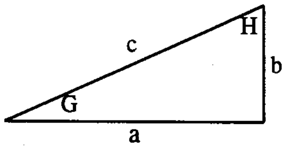
For each of the angles marked below (capital letters) write a triple:

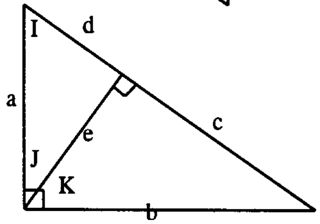








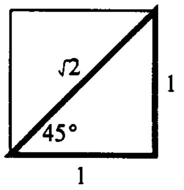




TRIPLES FOR 45°, 30° AND 60°

The angles 45°, 30° and 60° are simple angles which occur quite a lot and can easily be expressed with triples.

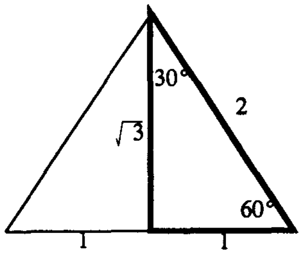
If you take a square of side 1 unit and draw a diagonal you will get a triangle with an angle of 45° because the diagonal cuts the right angle in half.



Pythagoras' theorem gives the diagonal length as $\sqrt{2}$ so a triple for 45° will be:

45°) 1, 1, $\sqrt{2}$

Similarly we can take an equilateral triangle of side 2 units and cut it in half:



The base of the bold triangle above is 1 unit as the equilateral triangle is cut in half.
For the same reason the 60° angle is cut in half to give 30° at the top.
And Pythagoras' theorem gives the height as $\sqrt{3}$.

4 From the diagram above write down a triple for 60° and a triple for 30°.
❋ Make a note of the triple for 45° as well.

GROUPS OF TRIPLES

If we look for the perfect triples all of whose elements are below 100 and whose angles are between 0° and 45° we obtain the following:

4	3	5	63	16	65
12	5	13	21	20	29
24	7	25	45	28	53
15	8	17	56	33	65
40	9	41	77	36	85
60	11	61	80	39	89
35	12	37	55	48	73
84	13	85	72	65	97

Studying these 16 triples we may observe three groups:

Group 1			**Group 2**			**Group 3**		
4	3	5	15	8	17	21	20	29
12	5	13	35	12	37	45	28	53
24	7	25	63	16	65	56	33	65
40	9	41				77	36	85
60	11	61				80	39	89
84	13	85				55	48	73
						72	65	97

Referring to the three elements in each triple by x, y, r we see in Group 1:

(a) y = the odd numbers in order from 3 onwards,
(b) $y^2 = x + r$,
(c) $r = x + 1$.

Using these patterns we can extend the list of triples in Group 1:
from (a) above, the next middle number must be 15,
from (b) $15^2 = 225$ = the sum of the outer numbers
and from (c) we split 225 into two consecutive numbers to get 112 and 113.

So the next entry in this list must be 112, 15, 113: which we may verify is also a triple.

Continuing we get: 112 15 113
144 17 145
etc.

5 What are the next two triples in this list.

In Group 2 we see: (a) y = every other even number starting at 8,
(b) $\frac{y^2}{2} = x + r$,
(c) $r = x + 2$.

Again we can extend the list of triples in Group 2:
from (a) the next middle number must be 20,
from (b) $\frac{20^2}{2} = 200$ = the sum of the outer numbers
and from (c) the outer numbers must be 99 and 101.

So the next triple is 99 20 101 and continuing:
143 24 145
etc.

6 What are the next two triples in this list.

Group 3 we leave for the moment as it has no obvious patterns.

CODE NUMBERS

We now bring groups 1 and 2 together:

TABLE 1

4	**3**	5
15	**4**×2	17
12	**5**	13
35	**6**×2	37
24	**7**	25
63	**8**×2	65
	etc.	

Here the triples in the two groups alternate and the even numbers in Group 2 are written as 4 x 2, 6 x 2 etc. so that the series of whole numbers from 3 onwards appears in a vertical column (in bold in TABLE 1).
The numbers in this column we call the code numbers of the triples: 3 is the code number for the triple 4, 3, 5; 4 is the code number for 15, 8, 17 and so on.

Thus each code number represents a triple and every triple in the two groups fits in and has its unique code number.
We will denote the code number in general by the letter c.

We see that for those triples in which the middle element is odd, that number is the triple code number; and for those in which the middle element is even the code number is half that even number.

7 Look carefully at the triples in Group 2 and predict the triple that precedes 15,8,17.
8 Look carefully at the triples in Group 1 and predict the triple that precedes 4,3,5.
9 How do these two triples fit into the sequence of triples in Table 1?

EXAMPLE 3

For the triple 684, 37, 685 the code number is 37, i.e. c = 37.

EXAMPLE 4

For the triple 1443, 76, 1445 the code number is 38, i.e. c = 38.

EXAMPLE 5

Find the triple with code number 19.

Since 19 is odd, square it to get 361 and find two consecutive numbers which add up to 361.
∴ the triple is 180, 19, 181.

EXAMPLE 6

Find the triple for which c = 22.

Since 22 is even, square it to get 484 and put down one less and one more than this for the outer elements of the triple. Then put double the code number for the middle element.
∴ the triple is 483, 44, 485.

EXERCISE 4

Write down the code numbers of the following triples:

1. 420, 29, 421 2. 1763, 84, 1765 3. 3120, 79, 3121 4. 4095, 128, 4097

Find the triples with the following code numbers: 5. 32 6. 37 7. 28 8. 101

46 Proof

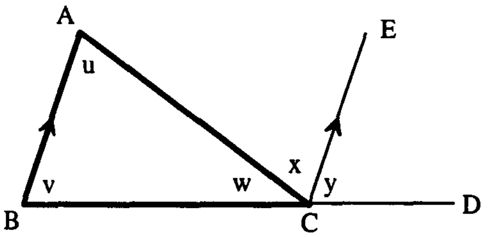
Proof is very important in mathematics. If we have a method for, say, squaring numbers that end in 5, we need to know that it always works- that there are no exceptions.

We found by measurement that the angles of a triangle add up to 180°. But measurement is not very accurate and if the total was 180.1° we would not know. We cannot measure all triangles either- until we have a proof we cannot be sure that there are no exceptions.

A proof is a convincing argument of the truth of some formula or method.

ANGLE SUM OF A TRIANGLE

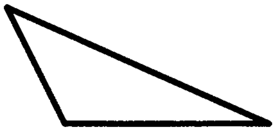
Here is a simple proof that the angles of any triangle add up to 180°.



Suppose we have a triangle ABC as shown above, with angles u, v and w.
We want to show that w+u+v = 180°.
We extend C to D and also draw CE parallel to BA.
Then **x=u** because they are Z-angles, and **y=v** because they are F-angles.
But **w+x+y = 180°**because they form a half circle.
Therefore since x=u and y=v, **w+u+v = 180°** which is what we wanted to prove.

You can also see from this argument that if the shape of the triangle changed it would still be valid.
It is important with proofs to understand their limitations. The proof above is only true for triangles, and only for flat triangles- it would not be true for triangles drawn on the surface of a sphere for example.

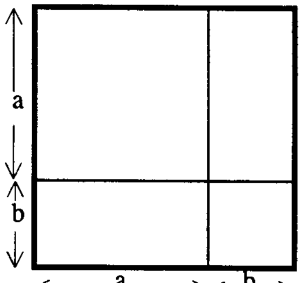
- Draw out a triangle with an obtuse angle like this:

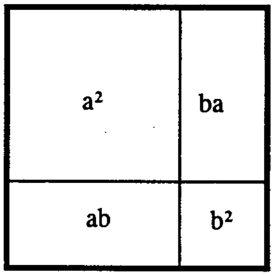


Extend the base and draw in the parallel line the same as in the proof above.
Check that the argument holds true for this obtuse-angled triangle also.

FIVE PROOFS USING AREAS

1. We have seen that $\mathbf{(a + b)^2 = a^2 + 2ab + b^2}$.
To prove this using areas we start with a square of side (a+b):

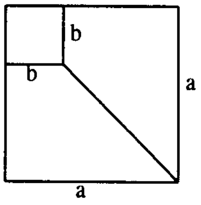


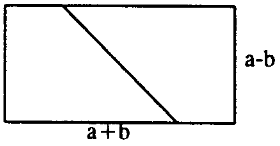


The left diagram shows the square with sides a+b and the right diagram shows the areas of each of the four parts.

$$\text{So } \mathbf{(a + b)^2} = a^2 + ba + ab + b^2 = \mathbf{a^2 + 2ab + b^2}.$$

2. We have seen that $\mathbf{a^2 - b^2 = (a + b)(a - b)}$. Here is a proof of this formula using areas:



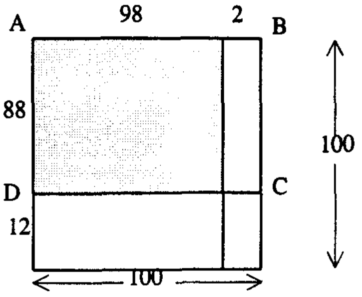


The larger square on the left has area a^2 and the smaller square has area b^2.
Then $a^2 - b^2$ is the area of the large square minus the area of the small square which leaves the two trapezia.
But these two trapezia can be rearranged as shown on the right above, to form a rectangle of base a+b and height a-b: which means its area is (a+b)(a-b).

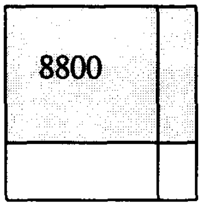
So $\mathbf{a^2 - b^2 = (a + b)(a - b)}$.

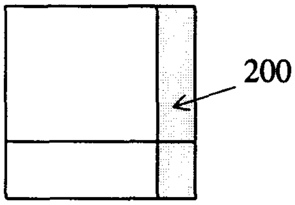
3. We have a very neat way of multiplying numbers together which are near a base number. For example to multiply 88 by 98 we note the deficiencies 12 and 2 and we write 88x98=8624 where 86=88-2 and 24=12X2.

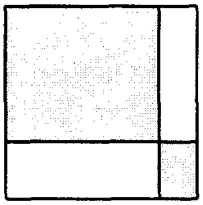
Let us look at a geometrical proof for this.
88×98 is the area of a rectangle 88 units by 98 units so we begin with a square of side 100:



You can see the required area shaded in the diagram.
You can also see the deficiencies from 100: 12 and 2.
Now the area ABCD must be 8800 because the base is 100 and the height is 88.







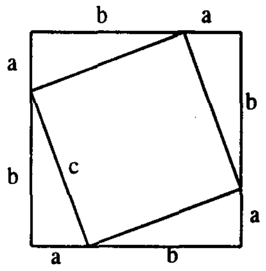
From this we subtract the strip on the right side, the area of which is 200: so 8800 - 200 = 8600. This leaves the required area but we have also subtracted the area of the small rectangle shown shaded above on the right. This must therefore be added back on and since its area is 12X2=24 we add 24 to 8600 to get 8624.

You can probably see that this procedure will work for any product when the numbers are close ti 100 and just below it.

1 Draw a diagram to show a similar proof for 96 X 97 = 9212.
2 Draw a diagram to show that 104 X 103 = 10712.

4. Pythagoras Theorem can also be proved using areas.

We begin with a square of side a+b in which we draw another square of side c:

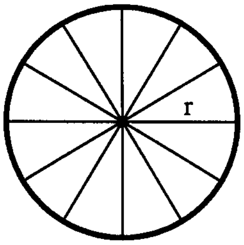


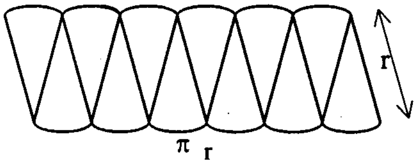
We then put the area of the large square equal to the area of the four triangles plus the area of the inner square: $(a + b)^2 = 4\times(\frac{1}{2}ab) + c^2$.

But $(a+b)^2 = a^2 + 2ab + b^2$ and $4\times(\frac{1}{2}ab) = 2ab$.
Therefore $a^2 + 2ab + b^2 = 2ab + c^2$ and so $\mathbf{a^2 + b^2 = c^2}$.

Since a, b and c are the sides of a right-angled triangle this proves Pythagoras' Theorem. ....

5. Area of a Circle. We have used the formula $\mathbf{A = \pi r^2}$ for finding the area of a circle Suppose we take a circle of radius r, cut it into 12 equal parts and rearrange the parts as shown on the right below:





The area of the shape on the right is the same as the area of the original circle.

The shape on the right is approximately a rectangle whose height is approximately r and whose base is approximately half the circumference of the circle.

But since $C = 2\pi r$, half the circumference will be πr.
So the area of the rectangle would be $r \times \pi r = \pi r^2$.

Now imagine that the number of parts into which the circle is cut increases, say to 100.
When the parts are rearranged the shape they make will be much close to a rectangle: the height o
this rectangle will be closer to r and the base will be closer to πr.

In fact the larger the number of parts the circle is divided into the closer the shape will be to a rectangle, the closer the height will be to r and the closer the base will be to πr.

So we can say that ultimately the shape becomes a rectangle the height becomes r and the base becomes πr. Which means that the area of the rectangle and therefore the area of the circle is ultimately equal to πr^2.

EVEN AND ODD NUMBERS

We have seen that 2n represents all even numbers because whatever whole number is chosen for n it will then get doubled which will make it even.
Or. to put it another way, if $n = 0, 1, 2, 3, 4 \ldots..$
$2n = 0, 2, 4, 6, 8 \ldots..$which are all even numbers.

Similarly $2n + 1$ represents all odd numbers because if 2n is always even then adding 1 to it will always make it odd.

Similarly $2n - 1$ will always be odd as well.
And if 2n is an even number then $2n + 2$ will be the next even number because even numbers are always 2 apart.

3 What will be the even number before 2n?

4 What will be the odd number after 2n+1?

EXAMPLE 1

Prove that the sum of two odd numbers is always even.

The odd numbers 13 and 25 add up to 38 which is an even number. Can we prove this will happen with any pair of odd numbers?

Let **2n+1** be the first odd number.
Let **2m+1** be another odd number.

We must use different letters, n and m, because the odd numbers must be different.

The sum of these is $\mathbf{2n+1 + 2m+1 = 2n + 2m + 2 = 2(n + m + 1)}$.

This proves the result because we have shown the sum of two odd numbers has a factor of 2 so it must therefore be even.

EXAMPLE 2

Show that the sum of three consecutive numbers is always a multiple of 3.

For example, $5 + 6 + 7 = 18$ and 18 is a multiple of 3.

If the first number is **n** the next number will be **n+1** and the third will be **n+2**.

Adding these gives $\mathbf{n + n+1 + n+2 = 3n + 3 = 3(n + 1)}$.
Which shows that the sum of three consecutive numbers is a multiple of 3.

EXAMPLE 3

Prove that the product of an odd number and an even number is always even.

7×10 is an example of an odd number multiplied by an even number and we can see the answer, 70, is even.

Let the odd number be **2n+1** and the even number be **2m**.
Then **(2n+1)2m** is the product and the factor 2 shows that it is always even.

In the following exercise begin by writing down a numerical example as you have seen in the examples above.

EXERCISE

Prove that:

a the product of two even numbers is even

b the sum of two even numbers is even

c the sum of four consecutive numbers is a multiple of 4

d the sum of two consecutive odd numbers is a multiple of 4

e the sum of three consecutive odd numbers is a multiple of 3

f the sum of three consecutive even numbers is a multiple of 6

REPRESENTING NUMBERS ALGEBRAICALLY

Sometimes it is useful to be able to represent 2 or 3-figure numbers algebraically.

EXERCISE

Find the value of **$10a+b$** when:

a $a=4, b=7$ **b** $a=3, b=1$ **c** $a=8, b=9$ **d** $a=1, b=6$

What are the values of a and b in $10a+b$ for the numbers:

e 72 **f** 63 **g** 75 **h** 99

This exercise shows that:

> The expression **$10a+b$** represents all 2-figure numbers where **a** is the left-hand figure and **b** is the right-hand figure.

We can use this to prove the method for squaring numbers that end in 5.
You will recall that to square a number, say 75, we multiply the number by the next number up: 8.
This gives the left part of the answer as 56 and the right part is always 25. So $75^2 = 5625$.

A 2-figure number that ends in 5 can be written as **$10a+5$**. (So for 75, $a=7$).
The square of such a number will therefore be $(10a+5)^2$.

And $\mathbf{(10a+5)^2 = 100a^2 + 100a + 25 = \underline{100a(a+1) + 25}}$.

This proves the method because a(a+l) is the left part of the answer (7X8 in our example). 100a(a+1) shows that a(a+l) is multiplied by 100. This is correct because the 56 we get in multiplying 7 by 8 is actually 5600 (that makes the 56 the left-hand part).

Finally 25 is added to 100a(a+1) which means that 25 is the right-hand part of the answer.

NIKHILAM MULTIPLICATION

We had a geometrical proof of the Nikhilam multiplication method earlier in this Chapter Here is an algebraic proof.

Numbers that are just below 100 can be represented by $(100 - a)$.
So for 94 which is 6 below 100, $a=6$ (a is the deficiency).
Similarly for 89, $a=11$.

Numbers just below 100 can be represented by **100 - a**.
Numbers just above 100 can be represented by **100 + a**.

Now if we want to multiply 2 **different** numbers together which are both just below 100 we can represent this by:

$$(100 - a)(100 - b).$$

Multiplying these brackets out we get $(100 - a)(100 - b) = 100^2 - 100a - 100b + ab$.

$$\text{So } (100 - a)(100 - b) = \underline{100(100 - a - b) + ab}. \qquad (1)$$

This actually proves the method as the following example shows.
If the values of a and b above are 12 and 2 then the sum being worked out is 88 × 98.

$$\text{So } a=12,\ 100 - a = 88,$$
$$b=2,\ 100 - b = 98.$$

In the result (1) above we see (100 - a - b) which is the first number, 100-a, minus the other deficiency, b. This is what we always do in Nikhilam multiplication: we take one of the numbers and take the other deficiency from it. In our example we find 88-2=86.

Then we see in (1) that this is multiplied by 100: 100(100 - a - b) .
As in the previous proof this puts the number in the left-hand part of the answer: 86 is actually 8600.

Finally (1) shows that ab is added on and this is just the product of the deficiencies. So in the example we have 8600 + 12X2 = 8624.

Similarly for two numbers which are both above 100 we get:

$$(100 + a)(100 + b) = 100(100 + a + b) + ab.$$

5 Show that the above equation is correct.

6 Using 102 X 107 as an example explain how result (2) proves the method for multiplying numbers just above 100.

7 These proofs are for multiplying numbers which are both above or both below 100.
Prove that the Nikhilam method works for multiplying numbers which are just below 1000.

PERFECT TRIPLES

The formula for generating perfect triples is: $\mathbf{c^2 - d^2}$, $\mathbf{2cd}$, $\mathbf{c^2 + d^2}$.
where c and d are integers.
We can prove that this always generates perfect triples by showing that the square of the first term plus the square of the second term equals the square of the third term.

I.e. show that $\mathbf{(c^2 - d^2)^2 + (2cd)^2 = (c^2 + d^2)^2}$.

$(c^2 - d^2)^2 + (2cd)^2 = c^4 - 2c^2d^2 + d^4 + 4c^2d^2 = c^4 + 2c^2d^2 + d^4$.

And $(c^2 + d^2)^2 = c^4 + 2c^2d^2 + d^4$. so they are equal and the above formula must always generate perfect triples.

● If you are not entirely convinced of this try to find values of c and d that do not produce a perfect triple.

8 If c, d are rational numbers (i.e. they can now be fractions, positive and negative as well as integers) does the proof still hold true?

9 If c, d can be rational numbers or square roots would it be true to say that the above formula always generates a perfect triples
b triples?

In the chapter on Triples we also saw how to get the code numbers c, d from the triple.
For example, for 12, 5, 13 we add the outer numbers to get c=25, and the middle number is d, d=5. This gives 25,5, which we cancel down to get 5,1.

To prove that this method gives the code numbers we can take the triple: $\mathbf{c^2 - d^2}$, $\mathbf{2cd}$, $\mathbf{c^2 + d^2}$ and apply the same procedure:
adding the outer terms we get $\mathbf{c^2 - d^2 + c^2 + d^2 = 2c^2}$.

And since the middle term is 2cd we get $\mathbf{2c^2 : 2cd = c : d}$ (cancelling by 2c).
The method gives us the code numbers c,d so it must work for all triples.

QUADRATIC EQUATIONS

Finally we will prove the formula for solving quadratic equations.

The formula we used was:

the differential is the square root of the discriminant.

We can put this into algebraic form by supposing that the quadratic equation is $\mathbf{ax^2 + bx + c = 0}$.

The differential would then be $\mathbf{2ax + b}$.
And the discriminant would be $\mathbf{b^2 - 4ac}$.

So we have to show that $\mathbf{2ax + b} = \pm\sqrt{b^2 - 4ac}$.

If we square both sides of this equation we get: $(2ax + b)^2 = b^2 - 4ac$.
Then squaring: $4a^2x^2 + 4abx + b^2 = b^2 - 4ac$.
Next subtract b^2 from each side: $4a^2x^2 + 4abx = -4ac$.

So $4a^2x^2 + 4abx + 4ac = 0$
Then dividing through by 4a gives: $\mathbf{ax^2 + bx + c = 0}$.

We can reverse each of these steps to get (3) by starting with $ax^2 + bx + c = 0$.

This proves the formula: differential = $\pm\sqrt{\text{discriminant}}$.

47 Coordinate Geometry

Coordinate geometry involves the use of coordinates to represent the positions of points and equations to represent lines and curves.
You have already done some work on this.

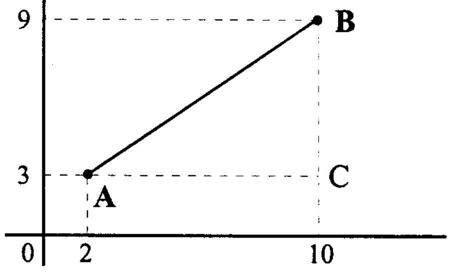
DISTANCE BETWEEN TWO POINTS

Given the coordinates of two points we may need to find the distance between the points.

EXAMPLE 1

Find the distance between A(2,3) and B(10,9).

We can first make a sketch to show what we know and what we want:



We want the distance AB.

Since AB is the hypotenuse of the right-angled triangle ABC we can find AB if we know AC and BC by using Pythagoras' theorem.

However from the diagram you will see that these lengths are known.
AC = 10-2 = 8 and BC = 9-3 = 6.

So $AB = 8^2 + 6^2 = 100$.
$\therefore$ AB = 10.

You will see that the side AC was found by subtracting the x-coordinates of the two points and BC was found by subtracting the y-coordinates.

EXAMPLE 2

Find the length of the line AB where A(-2,3), B(5,-4).

We subtract the x-coordinates: 5 - - 2 = 7.
We subtract the y-coordinates: -4 - 3 = -7.

We then square these results and add to get AB:

$AB = (7)^2 + (-7)^2 = 98.$
$\therefore \underline{AB = \sqrt{98}}.$

It does not matter if the coordinates are subtracted in the reverse order.
If we found the first coordinate minus the second we would get: -2 -5 = -7,
3 - -4 = 7.
Then when we square these and add we get 98, the same as before.

1 Draw a diagram like the one above to show the triangle with hypotenuse AB in Example 2.

EXERCISE

Find AB where A and B are the points:

a A(26,2), B(50,9)	**b** A(-5,9), B(4,-3)	**c** A(3,1), B(5,2)	**d** A(6,-10), B(30,0)
e A(-55,65), B(25,5)	**f** A(-1,7), B(-3,-2)	**g** A(21,10), B(-7,-35)	**h** A(23,-5), B(-2,-5)

2 Suppose the points were A(a,b) and B(c,d).
Complete the following formula for finding AB in terms of a, b, c and d.

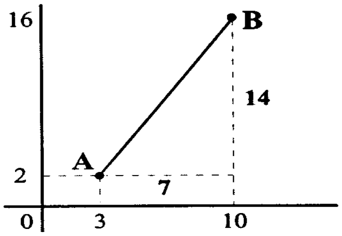
$$AB = (\text{--------})^2 + (\text{--------})^2.$$

GRADIENT OF A LINE JOINING TWO POINTS

EXAMPLE 3

Find the gradient of the line joining A(3,2) and B(10,16).

Let us begin by drawing a sketch:



As we did in the first section of this chapter we can form a right-angled triangle with AB as hypotenuse.

The base and height of this triangle can be found by subtracting the x and y coordinates of the given points. This gives 7 and 14 as shown.

You will recall that the gradient of a line is the vertical distance required to get back to the line after going 1 unit to the right of a point on the line.

In this case, since the height is clearly twice the base, if we go 1 unit to the right of A we need to go 2 units up to return to the line. So the gradient is 2.

This can also be found by simply dividing the height of the triangle by the base: $\frac{14}{7} = \underline{2}$.

This gives us a very simple method of finding the gradient of a line, given two points on it.

The **gradient** of a line joining two points is given by:

$$\frac{\text{the difference of the y-coordinates}}{\text{the difference of the x-coordinates}}$$

OR, if the points are (a,b), (c,d) then the gradient is

$$\frac{d-b}{c-a}$$

EXAMPLE 4

Find the gradient of the line joining the points (-3,4), (5,-2).

Following the method above the gradient is : $\frac{-2-4}{5--3} = \frac{-6}{8} = -\frac{3}{4}$.

- Look carefully at this example to make sure you agree with it.

It does not matter if we subtract the y-coordinates the other way around provided we subtract the x-coordinates in the same order. Had we started with the 4 above we would get $\frac{4--2}{-3-5} = \frac{6}{-8} = -\frac{3}{4}$, the same answer as before.

EXERCISE

Find the gradient of the line joining:

a (5,7) and (10,9)
b (1,7) and (15,11)
c (-2,6) and (3,-2)
d (4,0) and (-4,-1)
e (7,3) and (0,3)
f (5,-2) and (-2,5)
g (20,-1) and (-13,-8)
h (-6,-6) and (7,-5)
i (-9,-4) and (-30,-60)

EQUATION OF A LINE THROUGH TWO GIVEN POINTS

EXAMPLE 5

Find the equation of the line joining A(6,4) and B(4,1).

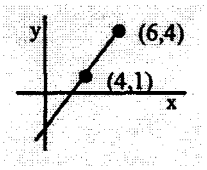
We write the coordinates one above the other as shown below:

$$\begin{array}{cc} 6 & 4 \\ 4 & 1 \\ \hline \end{array}$$

$$\underline{(6-4)y = (4-1)x + 6\times1 - 4\times4}$$

$$\underline{2y = 3x - 10}$$

The answer is 2y = 3x - 10 and this can be written straight down.



The coefficient of y is found by subtracting the x coefficients (6-4=2).
The coefficient of x is found by subtracting the y coefficients (4-1=3).

And the last term, -10, is found by cross-multiplying and subtracting (6×1-4×4=-10).

This is an application of the Sutra *Vertically and Crosswise* as we subtract vertically and then multiply cross-wise.

We can verify the answer is correct by showing that each of the points (6,4) and (4,1) lies on 2y=3x-10.

3 Show that (6, 4) and (4, 1) lie on the line 2y = 3x - 10.

EXAMPLE 6

Find the equation of the line joining (5, -4) and (3, 2).

Similarly:

$$\begin{array}{cc} 5 & -4 \\ \underline{3} & \underline{2} \\ \hline \multicolumn{2}{c}{\underline{2y = -6x + 22}} \end{array} \quad \text{or} \quad \underline{y = -3x + 11}.$$

In this example we divide through by 2 to simplify the answer.

EXERCISE

Find the equation of the line which passes through:

a (7,3) and (2,1)

b (8,-1) and (2, -5)

c (-1, 7) and (-5,4)

d (-3,0) and (1,6)

e (0,7) and (-7,0)

f (0,-3) and (-1,-2)

g (a,b) and (c,d)

4 Show that the points (a,b) and (c,d) lie on the line you obtained in question g above.

INTERSECTION OF TWO LINES

We expect two different straight lines to intersect- the only exception to this is when the lines are parallel, in which case they will never meet. Lines which are parallel have the same gradient so if two given lines have the same gradient we know there is no intersection.

We find the coordinates of the point of intersection of two lines by treating the equations of the lines as simultaneous equations.

Find the coordinates of the point of intersection of the lines $y = \frac{1}{2}x + 8$ and $y = -2x - 2$

The gradients are ½ and -2 so the lines do intersect.
We have seen how to solve simultaneous equations in an earlier chapter.
In this case we can equate the right-hand side of each equation and solve:

$$\frac{1}{2}x + 8 = -2x - 2,$$
$$2\frac{1}{2}x = -10,$$
$$x = -4.$$

Putting $x = -4$ into either of the two given equations gives $y = 6$.
So the required point is <u>(-4,6)</u>.

So we need to use the methods learnt earlier for solving pairs of equations.

EXAMPLE 8

Find the point of intersection of the lines $x + 2y = 5$ and $2x + 4y = -1$.

We see here that the gradient of both lines is $-\frac{1}{2}$ which means that the lines are parallel. There is therefore no point of intersection.

EXAMPLE 9

Find the point of intersection of the lines $2x + y = 7$ and $x - 2y = -9$.

In this case we can double the first equation and add it to the second in order to eliminate y:

$$\begin{array}{l} 4x + 2y = 14 \\ \underline{\ x - 2y = -9} \\ \underline{5x \qquad\ = 5} \quad \text{so } x = 1. \end{array}$$

Substituting into either equation gives $y = 5$ so the point is <u>(1,5)</u>.

- What happens if you try to solve the equations in Example 8?

EXERCISE

Find the coordinates of the point of intersection of the following pairs of lines (one of these cannot be done as the lines are parallel):

a $y = 2x - 3$, $y = -\frac{1}{2}x + 7$

b $y = x - 5$, $y = 3x + 5$

c $y = 2x + 7$, $y = 2x - 3$

d $x + y = 5$, $x - y = 9$

e $3x - 2y = 17$, $2x - 3y = 8$

f $x + 2y = 9$, $2x + 3y = 13$

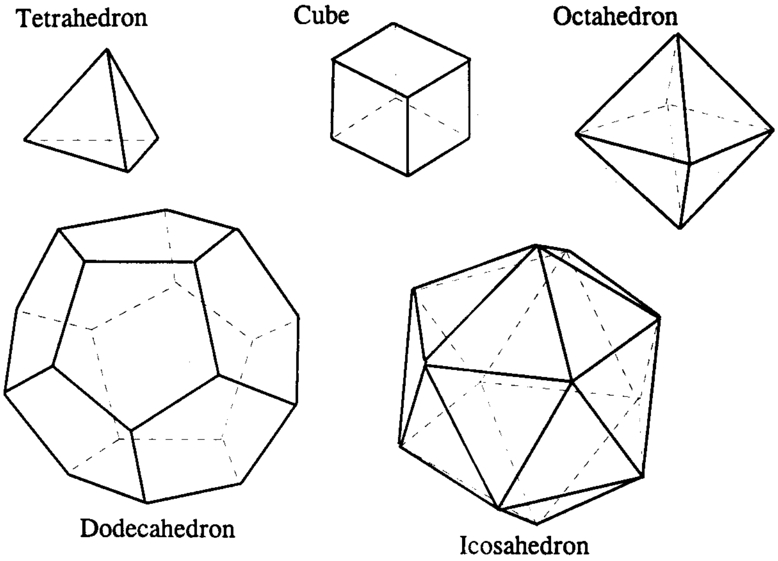
g The line through the points (2,-6) and (-2,2) intersects the line through the points (6,1) and (-2,-7) at X. Find the equation of each line and then find the coordinates of X.

THE PLATONIC SOLIDS

How many solid shapes can be made if all the faces of the shape have to be regular polygons and all the vertices have to be identical?

The answer is, only five. That there cannot be others was demonstrated by Euclid at the end of his thirteen books *The Elements*.

The five shapes are:



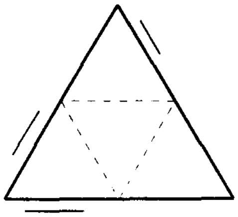
Pythagoras and Plato were very interested in these shapes and Plato related
to the five elements: Earth, Water, Fire, Air and Ether.
According to Plato the Cube represents Earth, the Icosahedron Water, the Tetrahedron Fire, the Octahedron Air and the Dodecahedron Ether.

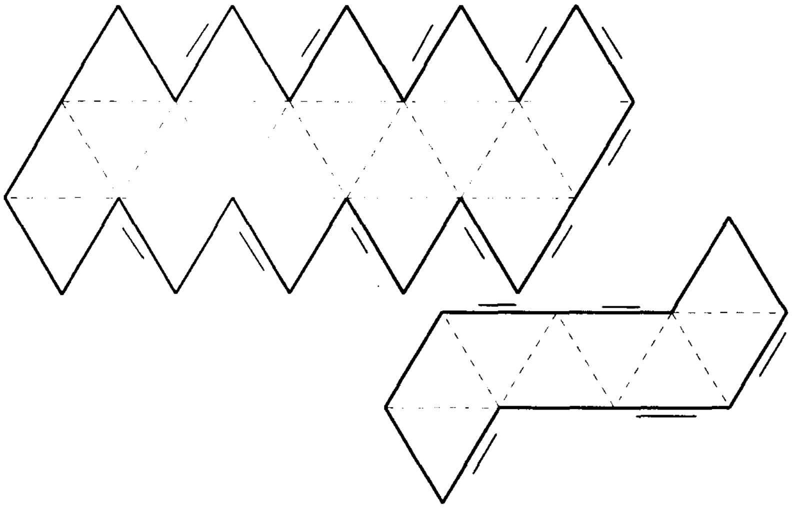
The Tetrahedron is composed of 4 equilateral triangles,
the Cube is composed of 6 squares,
the Octahedron is composed of 8 equilateral triangles,
the Dodecahedron is composed of 12 regular pentagons and
the Icosahedron is composed of 20 equilateral triangles.

- Draw the net of a cube on square paper and make yourself a cube (remember to put tabs on).

- Make the Tetrahedron, Octahedron and Icosahedron by drawing the nets given below on triangular paper. Decide the size for yourself. The short lines indicate where a tab is needed.

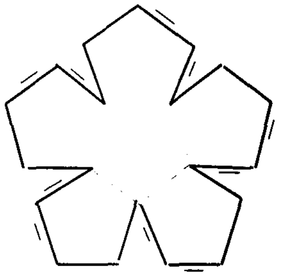
In each case one face has been left without any tabs: this face should be fixed last.





- Make a Dodecahedron using the net shown below.

You will need two of these which can then be fixed together. (Turn the second net over before attaching it to the first)
Use plain paper and cut out the regular pentagon on Worksheet 5 to use as a template.



These shapes have many fascinating properties.
The mid points of the faces of the Cube are the vertices of an Octahedron and the mid points of the faces of the Octahedron are the vertices of a Cube.

The mid points of the faces of the Icosahedron are the vertices of a Dodecahedron and the mid points of the faces of the Dodecahedron are the vertices of an Icosahedron.

The mid points of the faces of the Tetrahedron form another Tetrahedron.

The Icosahedron and the Dodecahedron also contain many golden rectangles.